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SCATTERING FOR THE K^--MESON FROM THE DEUTERON*

by

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ABSTRACT

Title of Thesis: Scattering of the K^--Meson from the Deuteron.

Anand Kumar Bhatia, Doctor of Philosophy, 1962.

Thesis directed by: Associate Professor Joseph Sucher.

A simple model is constructed to study the scattering of the K^--meson from the deuteron. This model does not treat the nucleon as heavy and takes into account the multiple scattering, the binding energy corrections and the contribution from the off-energy shell scattering. The scattering problem is investigated by using the Watson multiple scattering expansion of the transition-operator t. Considering the multiple scattering up to double order only, the t-matrix is written as

$$t = tp + tn + tc + t_I$$

where tp and tn corresponds to the single scattering of the K^--meson from the proton and the neutron in the deuteron. tc corresponds to the bound state contribution and t_I corresponds to the continuum state contribution of the K^--meson from the deuteron.

The interaction K^--P and the K^--N is taken as a point interaction and is of the form

$$tp = t_p^o \, \delta(r - \rho/2)$$
$$tn = t_n^o \, \delta(r + \rho/2)$$

where r and $\rho/2$ are the relative co-ordinates of the K^--meson and the nucleon in the center of mass system of the K^--d. t_p^o and t_n^o are taken as constants and are determined by using Dalitz solutions I and II given by Ross and Humphrey.

The potential between the nucleons in the deuteron is taken as a separable non-local potential which is such that the Hulthèn wave function satisfies the Schrödinger equation for the bound state of the deuteron.

The matrix elements corresponding to the bound state and the continuum state are calculated only for forward scattering. For any other scattering angle triple integrals are obtained and need too much numerical calculation.

The double scattering contribution is compared with the Brueckner model and also the correspondence between the two models is studied.

The forward differential, elastic and total cross sections are calculated.

The correction to the cross sections due to the charge exchange in the intermediate state is calculated.

The results of this model are compared with the experimental data of the K^--d scattering cross sections to study the predictions of this model and to find the favorable Dalitz solution.

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CHAPTER I.

INTRODUCTION

As neutron is not available in a free state, the scattering of the mesons from neutron has always been studied indirectly mainly through the study of the scattering of the meson from the deuteron. The deuteron is taken as the target because of its simple structure and also because it is a loosely bound system i.e. the average separation of its constituents is large compared to the range of the two-body interaction and its binding energy per particle is small. It would be expected that each of the nucleons in the deuteron would scatter the incident particle in a manner not much different from the way a free nucleon would scatter the incident particle.

The problems of nucleon-deuteron, pion-deuteron, and kaon-deuteron scattering are three-body problems; they have not been solved exactly. Aside from this basic fact, many of the physical details - spin dependence of nucleon-nucleon interaction, the presence of tensor forces and exchange forces in the nucleon-nucleon interaction - involved contribute to their complexity. Various approximations (the resonating group structure method (1), the Born approximation (2), the

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1 H.S. Massey, Prog. in Nucl. Phys., 3, 235 (1953).

2 T. Y. Wu and J. Ashkin, Phys. Rev., 373, 986 (1948).
G. F. Chew, ibid., 74, 809 (1948).
F. de Hoffman, ibid., 78, 216 (1950).
R. L. Gluckstern and H. A. Bethe, ibid., 81, 761 (1951).

high energy approximation[3], the impulse approximation[4], several variational procedures[5]) have been applied to these problems.

One of the approximations that has been applied to this kind of problem is the impulse approximation given by Chew[6] et al. In this approximation, the incident particle is viewed as scattering once from either of the two nucleons in the deuteron; each of these scatterings is viewed as the scattering from a free nucleon whose momentum distribution is that of the actual bound nucleon. The only role played by the intra-deuteron potential is the determination of this momentum distribution. This approximation does indeed lead to an expression (for example) for the elastic neutron-deuteron scattering cross section and a form factor for the deuteron structure. But this approximation neglects "potential" effects, multiple scattering effects, i. e. effects due to the incident particle's scattering more than once from the individual

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3 R. J. Glauber, "High Energy Collision Theory", in lectures in Theoretical Physics, edited by Wesley E. Brittin and Lita G. Danham (Interscience Publishers, Inc., New York, (1959), Vol. 1).

4 G. F. Chew, Phys.Rev., 84, 1057,(1951)
S. Fernback, T. A. Green and K. M. Watson, ibid., 84, 1084 (1951).
L. Castillego and L. S. Singh, Nuovo Cimento 11, 131 (1959).
Y. Sakamoto and T. Sasawaka, Prog. Theor. Phys. (Kyoto) 21, 879 (1959).
E. M. Ferreira, Phys. Rev., 115, 1727 (1959).

5 E. Clementel, Nuovo Cimento 8, 185 (1951).
L. Sartori, and S. I. Rubinow, Phys. Rev. 112, 714 (1958).
B. H. Bransden and R. G. Moorehouse, Nucl. Phys. 6, 310 (1958).
L. Spruch and L. Rosenberg, ibid., 17, 30 (1960).

6 G.F. Chew, Phys.Rev. 80, 196 (1950).
G. F. Chew and G. C. Wick, ibid., 85, 636 (1952).
G. F. Chew and M. L. Goldberger, ibid., 87, 778 (1952).

target nucleons, and the diminution in amplitude of the incident wave in crossing the nucleus.

(7)

The multiple scattering can be taken into account by Brueckner model. It is supposed in this model that the scattering taking place is S-wave scattering from the two point potentials. The wave function outside the potentials is written as the sum of the incident plane wave, a wave scattered from potential one, and a wave scattered from potential two i. e.

$$\Psi(\underline{r}) = e^{i\underline{q}\cdot\underline{r}} + A\,\frac{e^{iq|\underline{r}-\underline{R}_1|}}{|\underline{r}-\underline{R}_1|} + B\,\frac{e^{iq|\underline{r}-\underline{R}_2|}}{|\underline{r}-\underline{R}_2|} \qquad (1.1)$$

The outgoing amplitude A is given in terms of the total amplitude at R_1, by

$$A = \eta_p\left[e^{i\underline{q}\cdot\underline{R}_1} + B\,\frac{e^{iq\rho}}{\rho}\right], \text{ where } \rho = |\underline{R}_1 - \underline{R}_2| \qquad (1.2)$$

η_p is the K^--P scattering amplitude (in the deuteron).

Similarly

$$B = \eta_n\left[e^{i\underline{q}\cdot\underline{R}_2} + A\,\frac{e^{iq\rho}}{\rho}\right] \qquad (1.3)$$

η_n is the K^--N scattering amplitude (in the deuteron).

Solving for A and B from Eqs. (1.2) and (1.3) and substituting these values in Eq. (1.1), we get an expression for the scattering amplitude

$$F_f(\underline{q}',\underline{q}) = \left(1 + \eta_p\eta_n\frac{e^{2iq\rho}}{\rho^2}\right)^{-1}\left[\eta_p e^{i(\underline{q}-\underline{q}')\cdot\underline{R}_1} + \eta_n e^{i(\underline{q}-\underline{q}')\cdot\underline{R}_2} + \eta_p\eta_n\frac{e^{iq\rho}}{\rho}\left(e^{i(\underline{q}\cdot\underline{R}_1 - \underline{q}'\cdot\underline{R}_2)} + e^{i(\underline{q}\cdot\underline{R}_2 - \underline{q}'\cdot\underline{R}_1)}\right)\right] \qquad (1.4)$$

7 K.A. Brueckner, Phys.Rev., 89, 834 (1953); 90, 715 (1953).

The first two terms in the numerator of Eq. (1.4) are the single scattering (or impulse approximation) terms. The next two terms are the double scattering terms. The denominator represents all higher-order multiple scatterings. The propagator in the intermediate state is $e^{iq\rho}/\rho$ which is very large for the region $q\rho < 1$.

In this approximation, the nucleons are supposed to be infinitely heavy compared to the incident particle. Therefore, the recoil of the nucleons is neglected. This method has been used to study the pion-deuteron as well as the kaon-deuteron scattering. The validity of this method is questionable for pions and even more for heavy particles.

Drell and Verlet[8] consider the multiple-scattering corrections to the impulse approximation in the calculation of the scattering of $\pi^{\pm}$ mesons by deuterons.

They calculated the scattering amplitude for the meson-deuteron scattering problem by using t-matrix formalism. In this treatment the binding energy, motion of the sources and the absorption of the mesons are neglected. Three models are given:

(1) Brueckner model of point scatterer with propagator $e^{iq\rho}/\rho$. This model has the difficulties mentioned above.

(2) We consider the scatterers to be of finite extent and scattering from a separable potential of the type given by Yamaguchi[9].

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8 S. D. Drell and L. Verlet, Phys.Rev., 99, 849 (1955)

9 S. Yamaguchi, Phys.Rev., 95, 1628 (1954)

In this model, we approximate the Schrödinger equation describing the scattering of a particle by a potential, by replacing the wave function which appears in the interaction term by its average over the potential.

(3) The third model of momentum variation of the scattering amplitude considers only scattering on the energy shell in the intermediate state. In this approximation the intermediate wave propagates as $\frac{i \sin q\rho}{q\rho}$. This is smooth for $q\rho < 1$. Therefore it does not matter whether the sources are assumed to be of zero or of finite extent.

Using these models, it is found that the double scattering correction to the elastic cross section $\pi^{\pm} + D \rightarrow D + \pi^{\pm}$ is of the order of 10 percent or less and is model dependent. For the total cross sections (elastic, inelastic and absorption) as deduced from the imaginary part of the scattering amplitude in the forward direction, the correction is of the order of 10 percent or less and is quite model dependent.

Fulton and Schwed (10) have applied the Born and impulse approximations to calculate the nucleon-deuteron differential elastic cross sections. In order to carry out the impulse approximation calculation in complete detail, including, in particular, contributions from off-energy shell two particle matrix elements, the assumption is made that the two particle scattering is completely described by effective range theory. The results of the Born and impulse approximation differ

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10
Thomas Fulton and Philip Schwed, Phys. Rev., **115**, 973 (1959).

considerably. Their analysis shows that the experimental results are described better by the impulse approximation, including off-energy shell effects.

The Brueckner model has been used by Day, Snow and Sucher[11] to calculate the K^--D elastic and total cross sections. The calculations show that the impulse approximation does not give sensible results, and, therefore, the multiple scattering contribution is important.

One of the purposes of this thesis is to construct a simple model to study the scattering of the K^--meson from the deuteron, which does not treat the nucleon as heavy. This model takes into account the multiple scattering, the binding energy corrections and the contribution from the off-energy shell scattering. The scattering problem is investigated by using Watson[12] multiple scattering expansion of the transition-operator t. The expansion is given by

$$t = t_p + t_n + t_p \mathcal{G} t_n + t_n \mathcal{G} t_p + t_p \mathcal{G} t_n \mathcal{G} t_p + \cdots \qquad (1.5)$$

where

$$\mathcal{G} = (E - h - U + i\epsilon)^{-1}$$

$$t_p = t_{11} = V_1 + V_1 (E - h - U + i\epsilon)^{-1} t_1 \quad . \qquad (1.6)$$

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11 T.B.Day, G.A.Snow and J.Sucher, Nuovo Cimento, 14, 637 (1959).

12 K. M. Watson, Phys. Rev., 89, 575 (1953)

and

$$t_n = t_2 = V_2 + V_2 (E - h - U + i\epsilon)^{-1} t_2 \qquad (1.7)$$

where U is the potential between the nucleons in the deuteron. The other symbols are obvious.

In our calculation, we shall ignore the effects of the binding energy corrections on t_1 and t_2 due to the potential U.

The interaction between K^--meson and the nucleon is taken as a point interaction i. e.

$$t_1 = t_1^0 \, \delta(\underline{r} - \underline{\rho}/2) \qquad (1.8)$$

$$t_2 = t_2^0 \, \delta(\underline{r} + \underline{\rho}/2) \qquad (1.9)$$

where $\underline{r}$ is the meson position vector and $\underline{\rho} = \underline{r}_1 - \underline{r}_2$.

The t_1^0 and t_2^0 are taken as constants. No further approximations will be made in the calculations.

The magnitude of the t_1^0 and t_2^0 are determined with certain modifications by using the Dalitz[13] solutions of scattering lengths at low energies, defined by

$$q \cot \delta_I = 1/A_I \qquad (1.10)$$

where I = isotopic spin

For K^--P system I = 0, and 1

For K^--N system I = 1

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13 R. H. Dalitz and S. F. Tuan, Annals of Physics, 8, 100 (1959).

The scattering lengths are determined by the experimental data of the scattering of the K^--meson from the free nucleons. These solutions are given by Ross[14] and Humphrey[15].

The interaction between the nucleons in the deuteron is supposed to be given by a separable[16] non-local potential, which is such that in the case of the bound state of the deuteron, the Schrödinger equation is satisfied by the Hulthen wave function

$$\varphi(\rho) = N\left(e^{-\alpha\rho} - e^{-\beta\rho}\right)/\rho \tag{1.11}$$

where

$$N^2 = \alpha/2\pi \cdot \frac{1}{.606}$$

Using the potential deduced, the wave function for the continuum state is obtained by solving the Schrödinger equation.

The double scattering of the K^--meson is shown in the Feynmann diagram (Fig.1).

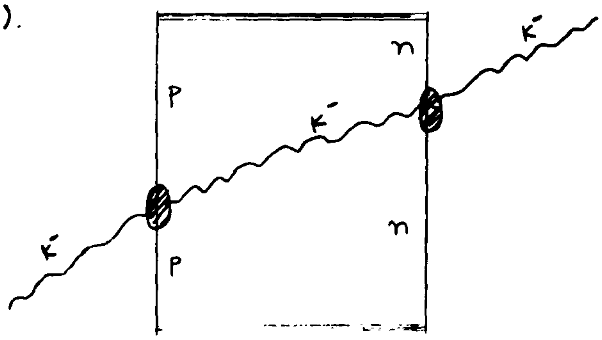

Figure 1.

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14 K. Ross, "Elastic and Charge-Exchange Scattering of K^- Mesons Incident on Hydrogen", Lawrence Radiation Lab. Rep., UCRL - 9749, June, 1961.

15 W. E. Humphrey, "Hyperon Production by K^- Meson Incident on Hydrogen", Lawrence Radiation Lab. Rep., UCRL - 9752, June, 1961.

16 See ref. 9.

The matrix elements corresponding to the bound state and the continuum state are calculated up to second order only for forward scattering and put in a suitable form for numerical calculations. For any other angle the matrix elements can be expressed in terms of triple integrals which are not very easy to handle numerically. The forward differential, elastic and total cross sections are evaluated at three energies. The total cross section is calculated using the optical theorem.

$$\sigma_T = 4\pi/q \ \mathrm{Im}\, f(0^\circ) \tag{1.12}$$

The correction to the elastic scattering due to charge exchange in the intermediate state is also calculated. The Feynmann diagram for this is shown in Fig. 2.

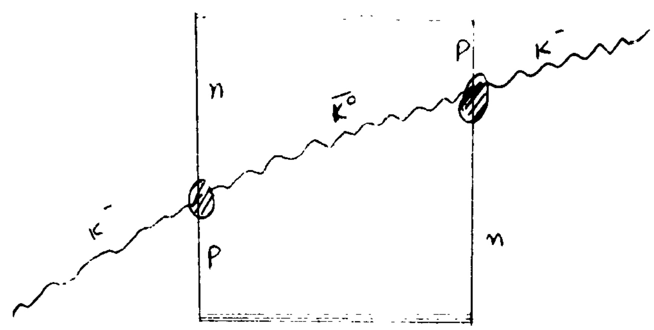

Figure 2.

The results of this model are compared with Brueckner model of S-wave scattering from two point potentials and also with the experimental data of the K^--D scattering cross sections to study the predictions of this model and to find the favorable Dalitz's solution at low energies.

CHAPTER II

K^--D SCATTERING

Let K^- be indicated by index '0', proton and neutron by '1' and '2'.

The initial and final wave functions for the K^--D system can be written as

$$\Psi_i = e^{i\underline{q}\cdot\underline{r}_0}\, e^{i\underline{K}\cdot\underline{R}}\, \phi(\underline{r}_1 - \underline{r}_2) \tag{2.1}$$

$$\Psi_f = e^{i\underline{q}'\cdot\underline{r}_0}\, e^{i\underline{K}'\cdot\underline{R}}\, \phi(\underline{r}_1 - \underline{r}_2) \tag{2.2}$$

where $\underline{q}$ and $\underline{q}'$ are the initial and final momenta of the K-meson; K and K' that of the deuteron.

$\underline{r}_0$, $\underline{r}_1$, $\underline{r}_2$ are the co-ordinates of the K-meson, proton and neutron, and

$$\underline{R} = \tfrac{1}{2}(\underline{r}_1 + \underline{r}_2)$$

Introduce,

$$\underline{\rho} = \underline{r}_1 - \underline{r}_2$$

$$\underline{r} = \underline{r}_0 - \underline{R} \tag{2.3}$$

$$\underline{S} = \frac{m_K \underline{r}_0 + m_d \underline{R}}{m_K + m_d} \qquad m_d = m_p + m_n$$

where m_K and m_d are the masses of the K^--meson and the deuteron

$$\underline{S} = \frac{m_K}{m_K + m_d}\,\underline{r} + \underline{R}$$

or

$$\underline{R} = \underline{S} - \frac{m_K}{m_K + m_d}\,\underline{r} \tag{2.4}$$

The T matrix for the K^--D scattering problem is given by

$$T = V + V(E - H + i\epsilon)^{-1} V \qquad (2.5)$$

where $V = V_1 + V_2$

V_1 is the potential between the K^--meson and the proton.

V_2 is the potential between the K^--meson and the neutron.

The Hamiltonian for the system is given by

$$H = h_0 + h_1 + h_2 + U + V \qquad (2.6)$$

where U is the potential between the proton and the neutron.

The matrix element for the scattering of the K^--meson from the deuteron is

$$\mathcal{M} = \langle \psi_f | T | \psi_i \rangle = \int e^{-i(\underline{q}'\cdot\underline{r}_0 + \underline{K}'\cdot\underline{R})} \varphi^*(\rho)\, T\, \varphi(\rho)\, e^{i(\underline{q}\cdot\underline{r}_0 + \underline{K}\cdot\underline{R})} d\underline{r}_0\, d\underline{r}_1\, d\underline{r}_2$$

Now $d\underline{r}_1 d\underline{r}_2 = d\rho\, d\underline{R}$

$$\mathcal{M} = \int e^{-i(\underline{q}'-\underline{q})\cdot\underline{r}_0}\, e^{-i(\underline{K}'-\underline{K})\cdot\underline{R}}\, \varphi^*(\rho) T \varphi(\rho)\, d\underline{r}_0\, d\rho\, d\underline{R}$$

Using Eqs. (2.3) and (2.4), we get

$$\begin{aligned}
\underline{q}\cdot\underline{r}_0 + \underline{K}\cdot\underline{R} &= \underline{q}\cdot(\underline{r} + \underline{R}) + \underline{K}\cdot\underline{R} \\
&= \underline{q}\cdot\underline{r} + (\underline{q} + \underline{K})\cdot\underline{R} \\
&= \underline{q}\cdot\underline{r} + (\underline{q} + \underline{K})\left(\underline{S} - \frac{m_K}{m_K + m_d}\underline{r}\right) \\
&= \left(\underline{q} - \frac{m_K}{m_K + m_d}(\underline{q} + \underline{K})\right)\cdot\underline{r} + (\underline{q} + \underline{K})\cdot\underline{S} \\
&= \left(\frac{m_d\,\underline{q} - m_K\,\underline{K}}{m_K + m_d}\right)\cdot\underline{r} + (\underline{q} + \underline{K})\cdot\underline{S} \\
&= \underline{q}_0\cdot\underline{r} + (\underline{q} + \underline{K})\cdot\underline{S}
\end{aligned}$$

$$\text{where } q_0 = \frac{m_d\,\underline{q} - m_K\,\underline{K}}{m_K + m_d} = \underline{q} - \frac{m_K(\underline{q} + \underline{K})}{m_K + m_d} \qquad (2.7)$$

$$\text{and} \quad d\underline{r}_0\, d\underline{R} = d\underline{r}\, d\underline{S} \qquad (2.8)$$

In the C. M. system of the K^--meson and the deuteron, the T matrix becomes

$$t = V + V\cdot\left[\frac{q_0^2}{2\mu_{Kd}} + \epsilon_B - h_r - h_d - U - V + i\epsilon\right]^{-1}\cdot V \quad (2.9)$$

where

$h_r = h_{relative} = -\nabla_r^2/2\mu_{Kd}$

$h_d = h_{deuteron} = -\nabla_d^2/2\mu_d$

$\epsilon_B =$ binding energy of the deuteron .

$\mu_{Kd} = \frac{m_K m_d}{m_K + m_d}$ is the reduced mass of the K^--meson and the deuteron.

$\mu_d = \frac{m_p m_n}{m_p + m_n}$ is the reduced mass of the deuteron.

Using Eqs. (2.7), (2.8) and (2.9), the matrix element becomes

$$\mathcal{M} = \int e^{-i(\underline{q}_0' - \underline{q}_0)\cdot \underline{r}} \cdot e^{-i(\underline{K}' - \underline{K} + \underline{q}' - \underline{q})\cdot \underline{S}} \varphi^*(\rho)\, t\, \varphi(\rho)\, d\rho\, d\underline{S}\, d\underline{r}$$

$$= (2\pi)^3 \delta(\underline{K}' - \underline{K} + \underline{q}' - \underline{q}) \cdot \int e^{-i(\underline{q}_0' - \underline{q}_0)\cdot \underline{r}} \varphi^*(\rho)\, t\, \varphi(\rho)\, d\underline{r}\, d\rho$$

$$= (2\pi)^3 \delta(\underline{K}' - \underline{K} + \underline{q}' - \underline{q}) \cdot \langle \underline{q}_0', \varphi \,|\, t \,|\, \varphi, \underline{q}_0 \rangle$$

$(2\pi)^3 \delta(\underline{K}' - \underline{K} + \underline{q}' - \underline{q})$ gives the conservation of momenta .

Matrix element in the center of mass is given by

$$M = \langle \underline{q}_0', \varphi | t | \varphi, q_0 \rangle \quad (2.10)$$

We can write

$$t = V + V\mathcal{G}V \quad (2.11)$$

where,

$$V = V_1 + V_2$$

and $\mathcal{G} = \left[\frac{q_0^2}{2\mu_{Kd}} + \epsilon_B - h_r - h_d - U - V + i\epsilon\right]^{-1}$

From the above equations the multiple expansion of t may be easily obtained in the form

$$t = \sum_{i=1}^{2} t_i + \sum_{i \neq j} t_i G t_j + \sum_{\substack{i \neq j \\ j \neq k}} t_i G t_j G t_k + \cdots \tag{2.12}$$

where t_i is defined by

$$t_i = V_i + V_i G t_i \tag{2.13}$$

and G is given by

$$G = \left[q_0^2/2\mu_{Kd} + \epsilon_B - h_K - h_d - U + i\epsilon \right]^{-1} \tag{2.14}$$

(17) We now separate t^d into a "coherent" part t_c corresponding to the process in which the deuteron is always in its ground state while the K^--meson is scattered from one nucleon to the other and a remaining part which will be called "incoherent". Thus we introduce $P = |\varphi\rangle\langle\varphi|$ the projection operator on the deuteron ground state and write

$$\begin{aligned} G &= GP + G(1-P) \\ &= G_0 P + G' \end{aligned} \tag{2.15}$$

where

$$G_0 = \left[q_0^2/2\mu_{Kd} - h_K + i\epsilon \right]^{-1} \tag{2.16}$$

and

$$(h_d + U)\varphi = \epsilon_B \varphi \tag{2.17}$$

On substituting Eq. (2.15) in Eq. (2.12) we get

$$t = t_1 + t_2 + t_c + t_I$$

where

$$t_c = \sum_{i \neq j} t_i G_0 P t_j \tag{2.18}$$

and

$$t_I = \sum_{i \neq j} t_i G' t_j \tag{2.19}$$

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[17] In the multiple expansion of t, we shall confine ourselves up to the double scattering. Therefore $t = t_1 + t_2 + t^d$ where t^d represents the term contributing to the double scattering only.

In this thesis we restrict our attention to the effects of single and double scattering only. In the next chapter we consider a model which allows us to compute the matrix elements of coherent part

$$t_1 G_0 P t_2 + t_2 G_0 P t_1 \tag{2.20}$$

and incoherent or continuum part,

$$t_1 G' t_2 + t_2 G' t_1 \tag{2.21}$$

in the form which allows numerical evaluation.

Using Eqs. (2.12), (2.15) in Eq. (2.10), we get

$$M = M_1 + M_2 + M_{12}^C + M_{12}^I \tag{2.22}$$

where

$$M_i = \langle \underline{q}_0', \Phi | t_i | \Phi, \underline{q}_0 \rangle \qquad i = 1,2 \tag{2.23}$$

$$M_{12}^C = \langle \underline{q}_0', \Phi | t_1 G_0 P t_2 | \Phi, \underline{q}_0 \rangle + (1 \leftrightarrow 2) \tag{2.24}$$

$$M_{12}^I = \langle \underline{q}_0', \Phi | t_1 G' t_2 | \Phi, \underline{q}_0 \rangle + (1 \leftrightarrow 2) \tag{2.25}$$

CHAPTER III

MODEL OF THE K⁻ MESON AND DEUTERON SCATTERING

A. Assumptions. 1. We shall suppose that the interaction between the K^--meson and the nucleon is a point interaction and therefore we can write[18]

$$t_p = t_1 = t_1^\circ \, \delta(r - \rho/2) \tag{3.1}$$

$$t_n = t_2 = t_2^\circ \, \delta(r + \rho/2) \tag{3.2}$$

where t_1° and t_2° are constants. (For the physical K^--meson scattering problem, t_1° and t_2° will be determined by the Dalitz solutions for the scattering lengths for zero effective range approximation for K^- scattered by free nucleon - See Chapter VII).

2. We shall suppose that the potential between the proton and the neutron in the deuteron is a separable non-local potential U which is such that the solution of the Schrödinger equation for bound state is the Hulthen wave function.

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18 This approximation corresponds to a K^--nucleon force range γ_0 short compared to the separation ρ of the nucleon in the deuteron and to the neglect of the binding interactions while the K^--meson interacts repeatedly with the same nucleon. This approximation should not be too bad for $\gamma_0 \approx 1f$ and $\rho \approx 5f$.

B. <u>Separable Potential and Ground State Wave Function of the Deuteron</u>. We introduce a separable non-local potential U which is such that the solution of Schrodinger equation

$$\left(p^2/2\mu_d + U\right)\varphi(\rho) = \epsilon_B \varphi(\rho) \tag{3.3}$$

for the deuteron in the ground state is the Hulthèn wave function

$$\varphi(\rho) = N \frac{\left(e^{-\alpha\rho} - e^{-\beta\rho}\right)}{\rho} \tag{3.4}$$

where $$N^2 = \frac{\alpha}{2\pi} \cdot \frac{1}{1-\alpha a} = \frac{\alpha}{2\pi} \cdot \frac{1}{.606} \tag{3.5}$$

and $$\beta = 6.2\,\alpha \tag{3.6}$$

$$\epsilon_B = -\frac{\alpha^2}{2\mu_d}$$ is the binding energy of the deuteron

Sch. equation becomes

$$\left(\frac{\alpha^2 - \nabla^2}{2\mu_d}\right)\varphi(\rho) = -U\varphi(\rho) \tag{3.7}$$

Assume

$$U\cdot\varphi(\rho) = -u(\rho)\int u(\rho')\,\varphi(\rho')\,d\rho' \tag{3.8}$$

Now

$$\nabla^2\varphi(\rho) = \frac{1}{\rho^2}\cdot\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right)\varphi(\rho)$$

$$\frac{\partial}{\partial\rho}\varphi(\rho) = N\left[\frac{e^{-\alpha\rho} - e^{-\beta\rho}}{-\rho^2} - \alpha\frac{e^{-\alpha\rho}}{\rho} + \beta\frac{e^{-\beta\rho}}{\rho}\right]$$

$$\left(\rho^2\frac{\partial}{\partial\rho}\varphi(\rho)\right) = N\left[-e^{-\alpha\rho} + e^{-\beta\rho} - \alpha\rho\, e^{-\alpha\rho} + \beta\rho\, e^{-\beta\rho}\right]$$

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\varphi(\rho)\right) = N\left[\alpha e^{-\alpha\rho} - \beta e^{-\beta\rho} - \alpha e^{-\alpha\rho} + \beta e^{-\beta\rho} + \alpha^2\rho\, e^{-\alpha\rho} - \beta^2\rho\, e^{-\beta\rho}\right]$$

$$\therefore \quad \nabla^2\varphi(\rho) = N\left(\frac{\alpha^2 e^{-\alpha\rho} - \beta^2 e^{-\beta\rho}}{\rho}\right)$$

Eq. (3.7) becomes

$$N\frac{\alpha^2}{2\mu_d}\left(\frac{e^{-\alpha\rho}-e^{-\beta\rho}}{\rho}\right) - \frac{N}{2\mu_d}\left(\frac{\alpha^2 e^{-\alpha\rho}-\beta^2 e^{-\beta\rho}}{\rho}\right) = +u(\rho)\int u(\rho')\varphi(\rho')\,d\rho'$$

or

$$\frac{N}{2\mu_d}(\beta^2-\alpha^2)\frac{e^{-\beta\rho}}{\rho} = u(\rho)\int u(\rho')\varphi(\rho')\,d\rho' \qquad (3.9)$$

Assume

$$u(\rho) = \frac{A}{\sqrt{2\mu_d}}\cdot\frac{e^{-\beta\rho}}{\rho} \qquad (3.10)$$

$$\text{R. H. S.} = \frac{N}{2\mu_d}\cdot A^2\frac{e^{-\beta\rho}}{\rho}\int_0^\infty\frac{e^{-\beta\rho'}}{\rho'}\left(\frac{e^{-\alpha\rho'}-e^{-\beta\rho'}}{\rho'}\right)\cdot 4\pi\rho'^2 d\rho'$$

$$= \frac{N}{2\mu_d}\cdot A^2\frac{e^{-\beta\rho}}{\rho}\cdot 4\pi\cdot\int_0^\infty\left(e^{-(\alpha+\beta)\rho'}-e^{-2\beta\rho'}\right)d\rho'$$

$$= \frac{N}{2\mu_d}\cdot 4\pi A^2\frac{e^{-\beta\rho}}{\rho}\cdot\left(\frac{1}{\alpha+\beta}-\frac{1}{2\beta}\right)$$

$$= \frac{N}{2\mu_d}\,4\pi A^2\frac{(\beta-\alpha)}{2\beta(\alpha+\beta)}\cdot\frac{e^{-\beta\rho}}{\rho}$$

Eq. (3.9) becomes

$$(\beta^2-\alpha^2)\frac{e^{-\beta\rho}}{\rho} = A^2\frac{e^{-\beta\rho}}{\rho}\cdot\frac{2\pi(\beta-\alpha)}{\beta(\alpha+\beta)}$$

$$\therefore\quad A^2 = \frac{\beta(\alpha+\beta)^2}{2\pi} \qquad (3.11)$$

$$\therefore \quad U(\rho,\rho') = -\frac{A^2}{2\mu\alpha}\cdot\frac{e^{-\beta\rho}}{\rho}\cdot\frac{e^{-\beta\rho'}}{\rho'}$$

$$= -\frac{\beta(\alpha+\beta)^2}{4\pi\mu\alpha}\cdot\frac{e^{-\beta\rho}}{\rho}\cdot\frac{e^{-\beta\rho'}}{\rho'} \tag{3.12}$$

Therefore

$$u(\rho) = \sqrt{\frac{\beta(\beta+\alpha)^2}{4\pi\mu\alpha}}\cdot\frac{e^{-\beta\rho}}{\rho} \tag{3.13}$$

Eq. (3.12) gives the potential in the deuteron in the co-ordinate space, we shall express the same potential in the momentum space by taking the Fourier transform of Eq. (3.13) .

Therefore, we get

$$u(p) = \frac{1}{(2\pi)^{3/2}}\int u(\rho)\, e^{-i\mathbf{p}\cdot\boldsymbol{\rho}}\, d\rho \tag{3.14}$$

$$= \sqrt{\frac{\beta(\alpha+\beta)^2}{4\pi\mu\alpha}}\cdot\frac{2\pi}{(2\pi)^{3/2}}\int_0^\infty\frac{(e^{ip\rho}-e^{-ip\rho})}{ip\rho}\,\frac{e^{-\beta\rho}}{\rho}\,\rho^2 d\rho$$

$$\doteq \sqrt{\frac{\beta(\alpha+\beta)^2}{4\pi\mu\alpha}}\cdot\frac{1}{(2\pi)^{1/2}}\,\frac{1}{ip}\left[\frac{1}{\beta-ip}-\frac{1}{\beta+ip}\right]$$

$$= \sqrt{\frac{\beta(\alpha+\beta)}{4\pi\mu\alpha}}\cdot\frac{2}{(2\pi)^{1/2}(p^2+\beta^2)} \tag{3.15}$$

In the momentum space we shall define

$$U(p,p') = -u(p)\cdot u(p') \tag{3.16}$$

From Eq. (3.15), we get

$$\begin{aligned} U(p,p') &= -\frac{\beta(\alpha+\beta)^2}{4\pi\mu\alpha}\cdot\frac{1}{2\pi}\cdot\frac{2}{p^2+\beta^2}\cdot\frac{2}{p'^2+\beta^2} \\ &= -\frac{\beta(\alpha+\beta)^2}{\pi^2}\cdot\frac{1}{2\mu\alpha}\cdot\frac{1}{p^2+\beta^2}\cdot\frac{1}{p'^2+\beta^2} \\ &= -\frac{B^2}{2\mu\alpha}\cdot\frac{1}{p^2+\beta^2}\cdot\frac{1}{p'^2+\beta^2} \end{aligned} \tag{3.17}$$

where

$$B^2 = \frac{\beta(\alpha+\beta)^2}{\pi^2} \tag{3.18}$$

C. <u>Continuum</u> <u>Wave</u> <u>Function</u>. Now we find the continuum solutions of the Schrödinger equation.

$$\psi_{\underline{k}}(\rho) = \varphi_{\underline{k}}(\rho) + G_0 U \psi_k(\rho) \tag{3.19}$$

where

$$\varphi_k(\rho) = \frac{e^{i\underline{k}\cdot\rho}}{(2\pi)^{3/2}} \tag{3.20}$$

and

$$G_0 = [E - H_0 + i\epsilon]^{-1} \tag{3.21}$$

Eq. (3.19) is equivalent to

$$(E - H_0 + i\epsilon)\,\psi_k(\rho) = U\psi_k(\rho)$$

Substituting Eq. (3.16) in the above equation in momentum space, we get

$$\left(\frac{k^2}{2\mu_1} - \frac{p^2}{2\mu_{\lambda}} + i\epsilon\right)\psi(p) - u(p)\int u(p')\psi(p')\,dp' \tag{3.22}$$

Using Eq. (3.17) in Eq. (3.22) and multiplying by ($2\mu_{\lambda}$) throughout, we get

$$(k^2 - p^2 + i\epsilon)\,\psi(p) = -\frac{\beta^2}{p^2+\beta^2}\cdot\int\frac{1}{p'^2+\beta^2}\cdot\psi(p')\,dp' \tag{3.23}$$

we shall suppose that

$$\psi(p) = \delta(\underline{p}-\underline{k}) - \frac{k^2+\beta^2}{2\pi^2}\cdot f(k)\cdot\frac{1}{p^2+\beta^2}\,\frac{1}{(k^2-p^2+i\epsilon)} \tag{3.24}$$

Substituting Eq. (3.24) in Eq. (3.23) and solving for $f(k)$, we find that

$$f(k) = \frac{1}{\frac{(k^2+\beta^2)^2}{\beta^2}\cdot\frac{1}{2\pi^2} - ik + \frac{k^2-\beta^2}{2\beta}} \tag{3.25}$$

In co-ordinate space

$$\psi(\rho) = \frac{1}{(2\pi)^{3/2}}\int\psi(p)\,e^{ip\cdot\rho}\,dp \tag{3.26}$$

Substituting Eq. (3.24) in Eq. (3.26), we get

$$\psi(\rho) = \frac{e^{i\underline{k}\cdot\rho}}{(2\pi)^{3/2}} - \frac{k^2+\beta^2}{2\pi^2} \cdot \frac{f(k)}{(2\pi)^{3/2}} \cdot \int \frac{1}{p^2+\beta^2} \cdot \frac{e^{i\underline{p}\cdot\rho}}{k^2-p^2+i\epsilon}\, d\underline{p}$$

$$= \frac{e^{i\underline{k}\cdot\rho}}{(2\pi)^{3/2}} - \frac{k^2+\beta^2}{2\pi^2} \cdot \frac{f(k)}{(2\pi)^{3/2}} \cdot I \tag{3.27}$$

where

$$I = \int \frac{1}{p^2+\beta^2} \cdot \frac{e^{i\underline{p}\cdot\rho}}{k^2-p^2+i\epsilon} \cdot d\underline{p}$$

$$= 2\pi \int_0^{\infty} \frac{e^{ip\rho} - e^{-ip\rho}}{ip\rho} \cdot \frac{1}{p^2+\beta^2} \cdot \frac{p^2\, dp}{(k^2-p^2+i\epsilon)}$$

$$= \frac{2\pi}{i\rho} \int_{-\infty}^{+\infty} e^{ip\rho} \cdot \frac{1}{p^2+\beta^2} \cdot \frac{p\, dp}{(k^2-p^2+i\epsilon)}$$

$$= \frac{2\pi}{i\rho} \cdot 2\pi i \times \text{(sum of residues)}$$

The poles are at $p = i\beta$, and $p = k + i\epsilon$

Residue at $p = i\beta$ is $\frac{e^{-\beta\rho}}{i\beta} \cdot \frac{i\beta}{(k^2+\beta^2)} = \frac{e^{-\beta\rho}}{2(k^2+\beta^2)}$

Residue at $p = k+i\epsilon$ is $\frac{-e^{+ik\rho}}{(k^2+\beta^2)} \cdot \frac{k}{2k} = -\frac{e^{+ik\rho}}{2(k^2+\beta^2)}$

$$I = \frac{2\pi^2}{(k^2+\beta^2)} \cdot \frac{e^{-\beta\rho} - e^{+ik\rho}}{\rho} \tag{3.28}$$

Substitute Eq. (3.28) in Eq. (3.27), we get (See ref. 9)

$$\psi_{\underline{k}}(\rho) = \frac{e^{i\underline{k}\cdot\rho}}{(2\pi)^{3/2}} - \frac{f(k)}{(2\pi)^{3/2}} \; \frac{k^2+\beta^2}{2\pi^2} \cdot \frac{2\pi^2}{k^2+\beta^2} \cdot \frac{e^{-\beta\rho} - e^{ik\rho}}{\rho}$$

$$= \frac{1}{(2\pi)^{3/2}} \left[e^{i\underline{k}\cdot\rho} + f(k) \cdot \left(e^{ik\rho} - e^{-\beta\rho} \right) \right] \qquad (3.29)$$

Eq. (3.29) is the required continuum wave function in the presence of the separable potential U.

In the next sections, we shall compute the single and double scattering of the K^--meson from the deuteron.

D. Single Scattering. For single scattering, the matrix element is

$M_1 = \langle \underline{q}', \varphi(\rho) | t_1 | \varphi(\rho), \underline{q} \rangle$ $\qquad$ $\underline{q}_1 = \underline{q}$ dropping the index .

Using Eq. (3.4), we get

$$M_1 = N^2 \iint e^{-i\underline{q}'\cdot\underline{r}}\, t_1\, e^{i\underline{q}\cdot\underline{r}} \left(\frac{e^{-\alpha\rho} - e^{-\beta\rho}}{\rho}\right)^2 d\rho\, d\underline{r}$$

Using Eq. (3.1), we get

$$M_1 = N^2 \iint e^{i(\underline{q}-\underline{q}')\cdot\underline{r}}\, t_1^{\circ}\, \delta(\underline{r} - \rho/2) \frac{(e^{-\alpha\rho} - e^{-\beta\rho})^2}{\rho^2}\, d\rho\, d\underline{r}$$

$$= N^2 t_1^{\circ} \int e^{i(\underline{q}-\underline{q}')\cdot\rho/2} \left(\frac{e^{-\alpha\rho} - e^{-\beta\rho}}{\rho}\right)^2 d\rho$$

$$= N^2 t_1^{\circ} \int_0^{\infty}\int_{-1}^{1} e^{iQ\rho x} \left(e^{-\alpha\rho} - e^{-\beta\rho}\right)^2 \cdot 2\pi\, d\rho\, dx .$$

$$= \frac{4\pi N^2 t_1^{\circ}}{Q} \int_0^{\infty} \frac{\sin Q\rho}{\rho} \cdot \left(e^{-\alpha\rho} - e^{-\beta\rho}\right)^2 d\rho$$

where $\qquad Q = \left|\frac{\underline{q} - \underline{q}'}{2}\right|$.

Using

$$\int_0^\infty \frac{\sin y}{y}\, e^{-Ay} dy = \tan^{-1}\left|1/A\right| \tag{3.30}$$

we get

$$M_1 = \frac{4\pi N^2 t_1^\circ}{Q}\int_0^\infty \frac{\sin y}{y}\left[e^{-\frac{2\alpha}{Q}y} + e^{-\frac{2\beta}{Q}y} - 2e^{-\frac{(\alpha+\beta)}{Q}y}\right]$$

$$= \frac{4\pi N^2 t_1^\circ}{Q}\left[\tan^{-1}\left|\frac{Q}{2\alpha}\right| + \tan^{-1}\left|\frac{Q}{2\beta}\right| - 2\tan^{-1}\left|\frac{Q}{\alpha+\beta}\right|\right] \tag{3.31}$$

where $y = Q\rho$

Similarly, we get

$$M_2 = \frac{4\pi N^2 t_2^\circ}{Q}\left[\tan^{-1}\left|\frac{Q}{2\alpha}\right| + \tan^{-1}\left|\frac{Q}{2\beta}\right| - 2\tan^{-1}\left|\frac{Q}{\alpha+\beta}\right|\right] \tag{3.32}$$

From Eqs. (3.31) and (3.32), we get

$$M_1 + M_2 = \frac{4\pi N^2 (t_1^\circ + t_2^\circ)}{Q}\left[\tan^{-1}\left|\frac{Q}{2\alpha}\right| + \tan^{-1}\left|\frac{Q}{2\beta}\right| - 2\tan^{-1}\left|\frac{Q}{\alpha+\beta}\right|\right] \tag{3.33}$$

$$Q = q \sin(\theta/2) \qquad \text{for} \qquad |\underline{q}| = |\underline{q}'| \tag{3.34}$$

$$M_1 + M_2 = (t_1^\circ + t_2^\circ) \qquad \text{for} \qquad \theta = 0^\circ \text{ or } q = 0 \tag{3.35}$$

E. <u>Double Scattering</u>: <u>Bound State or Coherent Scattering Contribution</u>. Using Eq. (2.20), coherent scattering contribution is given by

$$M_{12}^{c} = \left\langle \underline{q}', \varphi(\rho) \right| t_1 \left(q^2/2\mu_{\pi d} - h_{\pi} + i\epsilon \right)^{-1} P t_2 \left| \varphi(\rho'), \underline{q} \right\rangle + (1 \leftrightarrow 2) \tag{3.36}$$

Introducing $\sum_{\ell} |\underline{\ell}\rangle\langle\underline{\ell}| = \int \frac{d\underline{\ell}}{(2\pi)^3} |\underline{\ell}(\underline{r})\rangle \langle \ell(\underline{r}')|$

and $P = |\varphi(\rho)\rangle\langle\varphi(\rho')|$

in Eq. (3.36), we get in an obvious notation

$$M_{12}^{c} = \int \frac{d\underline{\ell}}{(2\pi)^3} \Big[\langle \underline{q}', \varphi(\rho) | t_1 | \varphi(\rho), \underline{\ell} \rangle \left(q^2/2\mu_{\pi d} - E_{\ell} + i\epsilon \right)^{-1} \times$$

$$\times \langle \underline{\ell}, \varphi(\rho) | t_2 | \varphi(\rho'), \underline{q} \rangle \Big] + (1 \leftrightarrow 2) . \tag{3.37}$$

Now

$$\langle \underline{q}', \varphi(\rho) | t_1 | \varphi(\rho), \underline{\ell} \rangle = \iint e^{-i\underline{q}'\cdot\underline{r}} |\varphi(\rho)|^2 \hat{t}_1 \, e^{i\underline{\ell}\cdot\underline{r}} \, d\underline{r} \, d\underline{\rho}$$

Using Eq. (3.1), we get

$$\langle \underline{q}', \varphi(\rho) | t_1 | \varphi(\rho), \underline{q} \rangle = \iint e^{i(\underline{\ell}-\underline{q}')\cdot\underline{r}} |\varphi(\rho)|^2 t_1^0 \delta(\underline{r} - \underline{\rho}/2) \cdot d\underline{r} \, d\underline{\rho}$$

$$= t_1^0 \int e^{i(\underline{\ell}-\underline{q}')\cdot\underline{\rho}/2} |\varphi(\rho)|^2 d\underline{\rho}$$

$$= t_1^0 \int_0^{\infty}\int_{-1}^{1} e^{i(\underline{\ell}-\underline{q}')\cdot\underline{\rho}/2} N^2 \left(e^{-\alpha\rho} - e^{-\beta\rho}\right)^2 d\rho \, 2\pi \, dx$$

$$= 4\pi N^2 t_1^0 \int_0^{\infty} \frac{\sin|(\underline{\ell}-\underline{q})\rho/2|}{|(\underline{\ell}-\underline{q})\rho/2|} \left(e^{-\alpha\rho} - e^{-\beta\rho}\right)^2 d\rho$$

Let

$$|\underline{l}-\underline{q}'|\rho/2 = y \quad , \quad d\rho = \frac{2}{|\underline{l}-\underline{q}'|}\,dy$$

$$\therefore \langle \underline{q}', \varphi(\rho)|t_1|\varphi(\rho), \underline{l}\rangle = \frac{8\pi N^2 t_1^0}{|\underline{l}-\underline{q}'|}\int_0^\infty dy\,\frac{\sin y}{y}\left[e^{-\frac{4\alpha y}{|\underline{l}-\underline{q}'|}} + e^{-\frac{4\beta y}{|\underline{l}-\underline{q}'|}} - 2e^{-\frac{2(\alpha+\beta)}{|\underline{l}-\underline{q}'|}y}\right]$$

Using Eq. (3.30), we get

$$\langle \underline{q}', \varphi(\rho)|t_1|\varphi(\rho), \underline{l}\rangle = \frac{8\pi N^2 t_1^0}{|\underline{l}-\underline{q}'|}\left[\tan^{-1}\left|\frac{\underline{l}-\underline{q}'}{4\alpha}\right| + \tan^{-1}\left|\frac{\underline{l}-\underline{q}'}{4\beta}\right| - 2\tan^{-1}\left|\frac{\underline{l}-\underline{q}'}{2(\alpha+\beta)}\right|\right]$$

Let

$$T(\underline{l}-\underline{q}) = \tan^{-1}\left|\frac{\underline{l}-\underline{q}}{4\alpha}\right| + \tan^{-1}\left|\frac{\underline{l}-\underline{q}}{4\beta}\right| - 2\tan^{-1}\left|\frac{\underline{l}-\underline{q}}{2(\alpha+\beta)}\right| \tag{3.38}$$

$$\langle \underline{q}', \varphi(\rho)|t_1|\varphi(\rho), \underline{l}\rangle = \frac{8\pi N^2 t_1^0}{|\underline{l}-\underline{q}'|}\,T(\underline{l}-\underline{q}') \tag{3.39}$$

Similarly

$$\langle \underline{l}, \varphi(\rho)|t_2|\varphi(\rho), \underline{q}\rangle = \frac{8\pi N^2 t_2^0}{|\underline{l}-\underline{q}|}\,T(\underline{l}-\underline{q}) \tag{3.40}$$

Substitute Eq. (3.39) and Eq. (3.40) in Eq. (3.37), we get

$$M_{12}^c = (8\pi N^2)^2 t_1^0 t_2^0 \int \frac{d\underline{l}}{(2\pi)^3}\cdot\frac{T(\underline{l}-\underline{q}')}{|\underline{l}-\underline{q}'|}\,\frac{1}{\left(\frac{q^2}{2\mu_{\pi d}} - \frac{l^2}{2\mu} + i\epsilon\right)}\,\frac{T(\underline{l}-\underline{q})}{(\underline{l}-\underline{q})} + (1\leftrightarrow 2)$$

$$= (8\pi N^2)^2 t_1^0 t_2^0 \cdot 4\mu_d \int \frac{d\underline{l}}{(2\pi)^3}\,\frac{T(\underline{l}-\underline{q}')}{|\underline{l}-\underline{q}'|}\,\frac{1}{(q^2 - l^2 + i\epsilon)}\cdot\frac{T(\underline{l}-\underline{q})}{|\underline{l}-\underline{q}|} \tag{3.41}$$

For any arbitrary angle between $\underline{q}$ and $\underline{q}'$, the above integral reduces to a triple integral which is not very convenient for numerical evaluation. Therefore, we shall evaluate it for the forward scattering only i.e. we shall take $\underline{q} = \underline{q}'$. Substituting $\underline{q} = \underline{q}'$ in Eq. (3.41), we get

$$M_{12}^{C} = \frac{(8\pi N^2)^2 t_1^0 t_2^0 \cdot 4\mu_{\kappa d}}{(2\pi)^3} \int_0^{\infty}\int_{-1}^{1} \frac{2\pi \ell^2 d\ell\, dx}{(q^2 - \ell^2 + i\epsilon)} \left(\frac{T(\underline{\ell}-\underline{q})}{|\underline{\ell}-\underline{q}|}\right)^2$$

$$= (64 N^4 t_1^0 t_2^0 \mu_{\kappa d}) \int_0^{\infty} \ell^2 d\ell \int_{-1}^{1} dx \frac{1}{(q^2 - \ell^2 + i\epsilon)} \cdot \left(\frac{T(\underline{\ell}-\underline{q})}{|\underline{\ell}-\underline{q}|}\right)^2$$

Let

$$|\underline{\ell} - \underline{q}| = \xi$$

or $$\ell^2 + q^2 - 2\ell q x = \xi^2$$

or $$\frac{dx}{\xi^2} = -\frac{1}{\ell q} \frac{d\xi}{\xi}$$

Therefore

$$M_{12}^{C} = (64 N^4 t_1^0 t_2^0 \mu_{\kappa d}) \int_0^{\infty} \ell^2 d\ell \int_{|\underline{\ell}+\underline{q}|}^{|\underline{\ell}-\underline{q}|} \frac{d\xi}{\xi} \frac{1}{(-\ell q)} \frac{1}{(q^2 - \ell^2 + i\epsilon)} (T(\xi))^2$$

$$= (64 N^4 t_1^0 t_2^0 \mu_{\kappa d}) \int_0^{\infty} \frac{\ell d\ell}{q} \int_{|\underline{\ell}-\underline{q}|}^{|\underline{\ell}+\underline{q}|} d\xi \frac{(T(\xi))^2}{\xi} \frac{1}{(q^2 - \ell^2 + i\epsilon)}$$

Let

$$q y = \ell$$

$$\therefore\ M_{12}^{c} = \left(64 N^4 t_1^{\circ} t_2^{\circ} \mu_{xd}\right) \int_0^{\infty} q^2 \frac{y\,dy}{q} \int_{q|1-y|}^{q|1+y|} d\xi \cdot \frac{(T(\xi))^2}{\xi} \cdot \frac{1}{q^2(1-y^2+i\epsilon)}$$

$$= \left(64 N^4 t_1^{\circ} t_2^{\circ} \mu_{xd}\right) \int_0^{\infty} y\,dy \int_{q|1-y|}^{q|1+y|} d\xi \frac{(T(\xi))^2}{\xi} \frac{1}{(1-y^2+i\epsilon)}$$

Now

$$\frac{1}{1-y^2+i\epsilon} = P\frac{1}{1-y^2} - i\pi\,\delta(1-y^2)$$

$$= P\frac{1}{1-y^2} - i\pi \frac{\delta(1-y)+\delta(1+y)}{2|y|}$$

We get

$$M_{12}^{c} = \left(\frac{64 N^4 t_1^{\circ} t_2^{\circ} \mu_{xd}}{q}\right)\Bigg[P\int_0^{\infty} y\,dy \int_{q|1-\xi|}^{q|1+\xi|} d\xi \frac{(T(\xi))^2}{\xi} \frac{1}{(1-y^2)}$$

$$- i\pi \int_0^{\infty} y\,dy \int_{q|1-y|}^{q|1+y|} d\xi \frac{(T(\xi))^2}{\xi} \frac{\delta(1-y)+\delta(1+y)}{2|y|}\Bigg]$$

$$= \left(\frac{64 N^4 t_1^{\circ} t_2^{\circ} \mu_{xd}}{q}\right)\Bigg[P\int_0^{\infty} y\,dy \int_{q|1-\xi|}^{q|1+\xi|} d\xi \frac{(T(\xi))^2}{\xi} \frac{1}{(1-y^2)}$$

$$- \frac{i\pi}{2} \int_0^{2q} d\xi \frac{(T(\xi))^2}{\xi}\Bigg]$$

Therefore,

$$M_{12}^{c} = \left(\frac{64N^4 t_1^\circ t_2^\circ \mu_{Kd}}{q}\right)\left[P\int_0^\infty \frac{y\,dy}{1-y^2}\int_{q|1-y|}^{q|1+y|} d\xi \, \frac{(T(\xi))^2}{\xi} - i\frac{\pi}{2}\int_0^{2q} d\xi \, \frac{(T(\xi))^2}{\xi}\right] \quad (3.42)$$

For $q = 0 = q'$, from Eq. (3.41), we get

$$M_{12}^{c} = -\frac{(8\pi N^2)^2 t_1^\circ t_2^\circ \, 4\mu_{Kd}}{(2\pi)^3}\int d\ell \, \left(\frac{T(\ell)}{\ell}\right)^2 \frac{1}{(\ell^2 - i\epsilon)}$$

$$= -\frac{(8\pi N^2)^2 t_1^\circ t_2^\circ \, 4\mu_{Kd} 4\pi}{(2\pi)^3}\int_0^\infty d\ell \, \frac{(T(\ell))^2}{\ell^2} \qquad \epsilon \to 0$$

$$= -(128 N^4 t_1^\circ t_2^\circ \mu_{Kd}) \int_0^\infty d\ell \, \frac{(T(\ell))^2}{\ell^2} \quad (3.43)$$

where

$$T(\ell) = \tan^{-1}\left|\frac{\ell}{4\alpha}\right| + \tan^{-1}\left|\frac{\ell}{4\beta}\right| - 2\tan^{-1}\left|\frac{\ell}{2(\alpha+\beta)}\right| \quad (3.44)$$

F. Double Scattering: Continuum State Scattering Contribution. The double scattering in case of the continuum state is given by

$$M_{12}^{I} = \langle \underline{q}', \varphi | t_1 \left(q^2/2\mu_{\kappa d} + \epsilon_B - h_r - h_d + U + i\epsilon \right)^{-1} \times$$

$$\times t_2 | \varphi, \underline{q} \rangle + (1 \leftrightarrow 2) \qquad (3.45)$$

We shall use Eq. (3.29) for the continuum state of the deuteron in the intermediate state. Since the Eq. (3.29) includes the effects of the potential U of the nucleon in the deuteron, we have

$$\sum_{\underline{\ell}, \underline{t}} | \underline{\ell}, \psi_{\underline{t}} \rangle \langle \underline{\ell}, \psi_{\underline{t}} | + \sum_{\ell} | \underline{\ell}, \varphi \rangle \langle \underline{\ell}, \varphi | = 1 \qquad (3.46)$$

and $\langle \psi_{\underline{t}} | (1 - P) = \langle \psi_t | \quad ; \quad \langle \varphi | (1 - P) = 0 \qquad (3.47)$

We get, on introducing

$$\sum_{\ell} | \underline{\ell} \rangle \langle \underline{\ell} | = \int \frac{d\underline{\ell}}{(2\pi)^3} | \ell(\underline{r}) \rangle \langle \ell(\underline{r}') |$$

and $$\sum_{t} | \psi_{\underline{t}} \rangle \langle \psi_{\underline{t}} | = \int d\underline{t} \; | \psi_{\underline{t}}(\underline{\rho}) \rangle \langle \psi_{\underline{t}}(\underline{\rho}') |$$

in Eq. (3.45),

$$M_{12}^{I} = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^3} \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \psi_{\underline{t}} \rangle \left(q^2/2\mu_{\kappa d} + \epsilon_B - E_\ell - E_t + i\epsilon \right)^{-1}$$

$$\times \langle \psi_{\underline{t}}, \underline{\ell} | t_2 | \varphi, \underline{q} \rangle \quad + (1 \leftrightarrow 2) \qquad (3.48)$$

where

$$E_\ell = \ell^2/2\mu_{\kappa d} \qquad (3.49)$$

and $$E_t = t^2/2\mu_d \qquad (3.50)$$

we can write Eq. (3.29) as

$$\psi_t(\rho) = \frac{1}{(2\pi)^{3/2}}\left(|t;\rho\rangle + \chi_t(\rho)\right) \tag{3.51}$$

where

$$\chi_t(\rho) = f(t)\left(\frac{e^{it\rho} - e^{-\beta\rho}}{\rho}\right) \tag{3.52}$$

and

$$f(t) = \frac{1}{\frac{(t^2+\beta^2)^2}{2\beta(\alpha+\beta)^2} - it + \frac{t^2-\beta^2}{2\beta}} \tag{3.53}$$

Substituting Eq. (3.51) in Eq. (3.48), we get

$$M_{12}^{I} = {}^{(1)}M_{12}^{I} + {}^{(2)}M_{12}^{I} + {}^{(3)}M_{12}^{I} + {}^{(4)}M_{12}^{I} + (1 \leftrightarrow 2)$$

where

$${}^{(1)}M_{12}^{I} = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^6} \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \underline{t}\rangle \left(\frac{q^2}{2\mu_{\kappa d}} + \epsilon_B - E_\ell - E_t + i\epsilon\right)^{-1} \langle \underline{t}, \underline{\ell} | t_2 | \varphi, \underline{q}\rangle \tag{3.54}$$

$${}^{(2)}M_{12}^{I} = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^6} \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \chi_t\rangle \left(\frac{q^2}{2\mu_{\kappa d}} + \epsilon_B - E_\ell - E_t + i\epsilon\right)^{-1} \langle \underline{t}, \underline{\ell} | t_2 | \varphi, \underline{q}\rangle \tag{3.55}$$

$${}^{(3)}M_{12}^{I} = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^6} \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \underline{t}\rangle \left(\frac{q^2}{2\mu_{\kappa d}} + \epsilon_B - E_\ell - E_t + i\epsilon\right)^{-1} \langle \chi_t, \underline{\ell} | t_2 | \varphi, \underline{q}\rangle \tag{3.56}$$

$${}^{(4)}M_{12}^{I} = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^6} \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \chi_t\rangle \left(\frac{q^2}{2\mu_{\kappa d}} + \epsilon_B - E_\ell - E_t + i\epsilon\right)^{-1} \langle \chi_t, \underline{\ell} | t_2 | \varphi, \underline{q}\rangle \tag{3.57}$$

1. __Computation of__ $\overset{(1)}{M}{}^{I}_{12}$. We shall compute the matrix element $\overset{(1)}{M}{}^{I}_{12}$ first.

$$\overset{(1)}{M}{}^{I}_{12} = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^6} \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \underline{t} \rangle \left(q^2/2\mu_{\alpha d} + \epsilon_B - E_\ell - E_t + i\epsilon\right)^{-1} \langle \underline{t}, \underline{\ell} | t_2 | \varphi, \underline{q} \rangle \quad (3.54)$$

The Fourier transform of Eq. (3.4) is

$$\varphi(\rho) = \bar{N} \int e^{i\underline{p}\cdot\rho} \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p^2+\beta^2} \right) \quad (3.58)$$

where

$$\bar{N} = N/2\pi \quad (3.58')$$

Now

$$\begin{aligned}
\langle \underline{q}', \varphi | t_1 | \underline{\ell}, \underline{t} \rangle &= \iint d\rho\, d\underline{r}\; e^{-i\underline{q}'\cdot\underline{r}}\, \varphi^*(\rho)\, t_1\, e^{i\underline{\ell}\cdot\underline{r}}\, e^{i\underline{t}\cdot\rho} \\
&= t_1^0 \iint d\rho\, d\underline{r}\; e^{i(\underline{\ell}-\underline{q}')\cdot\underline{r}}\, \varphi^*(\rho)\, \delta(\underline{r}-\rho/2)\, e^{i\underline{t}\cdot\rho} \\
&= t_1^0 \int d\rho\; e^{i(\underline{t}+\frac{1}{2}(\underline{\ell}-\underline{q}'))\cdot\rho}\, \varphi^*(\rho)
\end{aligned}$$

Using Eq. (3.58), we get

$$\begin{aligned}
\langle \underline{q}', \varphi | t_1 | \underline{\ell}, \underline{t} \rangle &= \bar{N} t_1^0 \iint d\rho\, d\underline{p}'\, e^{-i\underline{p}'\rho}\, e^{i(\underline{t}+\frac{\underline{\ell}-\underline{q}}{2})\cdot\rho} \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p^2+\beta^2} \right) \\
&= \bar{N} t_1^0 \iint d\rho\, d\underline{p}'\, e^{i(-\underline{p}'+\underline{t}+\frac{1}{2}(\underline{\ell}-\underline{q}'))\cdot\rho} \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p'^2+\beta^2} \right) \\
&= \bar{N} t_1^0 \int d\underline{p}'\, (2\pi)^3 \delta(-\underline{p}'+\underline{t}+\tfrac{1}{2}(\underline{\ell}-\underline{q})) \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p^2+\beta^2} \right) \\
&= \bar{N} t_1^0\, (2\pi)^3 \left(\frac{1}{p'^2+\alpha^2} - \frac{1}{p'^2+\beta^2} \right) \quad (3.59)
\end{aligned}$$

where

$$\underline{p}' = \underline{t} + \frac{1}{2}(\underline{\ell}-\underline{q}') \quad (3.59')$$

Similarly

$$\langle \underline{\ell}, \underline{t} \,|\, t_2 \,|\, \varphi, \underline{q} \rangle = (2\pi)^3 \bar{N}\, t_2^\circ \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p^2+\beta^2} \right) \tag{3.60}$$

where

$$\underline{p} = \underline{t} - \tfrac{1}{2} \cdot (\underline{\ell} - \underline{q}) \tag{3.60'}$$

Using Eqs. (3.59) and (3.60) in Eq. (3.54), we get

$$\begin{aligned} {}^{(1)}M_{12}^{I} &= (\bar{N})^2 t_1^\circ t_2^\circ \iint d\underline{\ell}\, d\underline{t} \left[\left(\frac{1}{p'^2+\alpha^2} - \frac{1}{p'^2+\beta^2} \right) \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p^2+\beta^2} \right) \times \right. \\ &\qquad \left. \times \left(q^2/2\mu_{Kd} + \epsilon_B - E_\ell - E_t + i\epsilon \right)^{-1} \right] \\ &= (\bar{N})^2 t_1^\circ t_2^\circ \iint d\underline{\ell}\, d\underline{t} \left[\left(\frac{1}{p'^2+\alpha^2} - \frac{1}{p'^2+\beta^2} \right) \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p^2+\beta^2} \right) \times \right. \\ &\qquad \left. \times (-2\mu_d) \left(t^2 - \frac{\mu_d}{\mu_{Kd}} (q^2 - \ell^2) + \alpha^2 - i\epsilon \right)^{-1} \right] \end{aligned} \tag{3.61}$$

since $\quad \epsilon_B = - \alpha^2/2\mu_d \quad .$

We shall put[18]

$${}^{(1)}M_{12}^{I} = M_{12}^{I}(\alpha\alpha) + M_{12}^{I}(\beta\beta) - M_{12}^{I}(\alpha\beta) - M_{12}^{I}(\beta\alpha) \tag{3.62}$$

where

$$\begin{aligned} M_{12}^{I}(AB) &= (\bar{N})^2 t_1^\circ t_2^\circ (-2\mu_d) \iint d\underline{\ell}\, d\underline{t} \left[\left(p'^2 + A^2 \right)^{-1} \cdot \left(p^2 + B^2 \right)^{-1} \times \right. \\ &\qquad \left. \times \left(t^2 - \frac{\mu_d}{\mu_{Kd}} (q^2 - \ell^2) + \alpha^2 - i\epsilon \right)^{-1} \right] \end{aligned} \tag{3.63}$$

---

18 This separation is only for computational convenience and introduces spurious divergences which are cancelled at the end.

We combine denominators in order to make the denominator a function of t^2 and ℓ^2 only, by using the Feynmann formula

$$\frac{1}{\bar{a}\bar{\ell}\bar{c}} = \int_0^1\int_0^1 \frac{2u\,du\,dv}{\left[\bar{a}(1-u) + \bar{\ell}uv + \bar{c}u(1-v)\right]^3} \tag{3.64}$$

Let

$$\bar{a} = t^2 - \frac{\mu_d}{\mu_{Kd}}(q^2 - \ell^2) + \alpha^2 - i\epsilon$$

$$\bar{\ell} = t^2 + \underline{t}\cdot(\underline{\ell} - \underline{q}') + \left(\frac{\underline{\ell}-\underline{q}'}{2}\right)^2 + A^2$$

$$\bar{c} = t^2 - \underline{t}\cdot(\underline{\ell} - \underline{q}) + \left(\frac{\underline{\ell}-\underline{q}}{2}\right)^2 + B^2$$

we can write

$$\bar{a}(1-u) + \bar{\ell}uv + \bar{c}u(1-v) = (\underline{t} + \underline{F})^2 + D^2$$

where

$$\underline{F} = \tfrac{1}{2}\left[(\underline{\ell}-\underline{q}')uv - (\underline{\ell}-\underline{q})u(1-v)\right]$$

$$D = -\tfrac{1}{4}\left[(\underline{\ell}-\underline{q}')uv - (\underline{\ell}-\underline{q})u(1-v)\right]^2 + \left(\tfrac{1}{4}\cdot(\underline{\ell}-\underline{q}')^2 + A^2\right)uv$$

$$+ \left(\tfrac{1}{4}(\underline{\ell}-\underline{q})^2 + B^2\right)u(1-v) - \left(\frac{\mu_d}{\mu}(q^2-\ell^2) + i\epsilon\right)(1-u) + \alpha^2(1-u)$$

or

$$D = c\ell^2 + \underline{\ell}\cdot\underline{b} + a$$

$$= c\left(\ell - \frac{b}{2c}\right)^2 + \frac{4ac - b^2}{4c} \tag{3.65}$$

where

$$a = -(q')^2\frac{u^2v^2}{4} - q^2\frac{u^2(1-v)^2}{4} + 2\,\underline{q}\cdot\underline{q}'\,u^2\frac{v(1-v)}{4}$$

$$+\left(\frac{(q')^2}{4} + A^2\right)uv + \left(\frac{q^2}{4} + B^2\right)u(1-v) - \frac{\mu_d}{\mu_{Kd}}q^2(1-u)$$

$$+ \alpha^2(1-u) - i\epsilon(1-u) \quad (3.66)$$

$$\underline{b} = \left(\underline{q}'\left(v^2 - \frac{v}{2}\right) + \underline{q}\left(\tfrac{1}{2} - \tfrac{3v}{2} + v^2\right)\right)u^2 + \left((\underline{q} - \underline{q}')\tfrac{v}{2} - \tfrac{\underline{q}}{2}\right)u \quad (3.67)$$

$$c = u^2\left(v - v^2 - \tfrac{1}{4}\right) + \frac{u}{4}\left(1 - 4\mu_d/\mu_{Kd}\right) + \mu_d/\mu_{Kd} \quad (3.68)$$

$$\therefore\ M_{12}^{I}(AB) = \bar{N}^2 t_1^\circ t_2^\circ(-2\mu_d)\iint d\underline{t}\,d\underline{\ell}\int_0^1\int_0^1 \frac{2u\,du\,dv}{\left[(\underline{t} + \underline{E})^2 + D^2\right]^3}$$

Making the transformation $\underline{t} \to \underline{t} - \underline{E}$, we get

$$M_{12}^{I}(AB) = \bar{N}^2 t_1^\circ t_2^\circ(-2\mu_d)\iint d\underline{t}\,d\underline{\ell}\int_0^1\int_0^1 \frac{2u\,du\,dv}{(\underline{t}^2 + D^2)^3}$$

Substituting for D^2 from Eq. (3.65), we get

$$M_{12}^{I}(AB) = \bar{N}^2 t_1^\circ t_2^\circ(-2\mu_d)\int_0^1\int_0^1 2u\,du\,dv\iint\frac{d\underline{t}\,d\underline{\ell}}{\left(\underline{t}^2 + c(\underline{\ell} - \underline{b}/2c)^2 + \frac{4ac - b^2}{4c}\right)^3}$$

Next, make the transformation

$$\sqrt{c}\,(\underline{\ell} - \underline{b}/2c) \to \underline{\ell}\ .$$

We get

$$M_{12}^{I}(AB) = \bar{N}^2 t_1^\circ t_2^\circ(-2\mu_d)\int_0^1\int_0^1\frac{2u\,du\,dv}{c^{3/2}}\iint\frac{d\underline{t}\,d\underline{\ell}}{\left(t^2 + \ell^2 + \frac{4ac - b^2}{4c}\right)^3} \quad (3.69)$$

We introduce the six dimensional space given by

$$\underline{R}^2 = \underline{t}^2 + \underline{\ell}^2 \tag{3.70}$$

so that

$$\begin{aligned} d\underline{t}\, d\underline{\ell} &= d^6\underline{R} \\ &= R^5 dR\, d\Omega \end{aligned} \tag{3.71}$$

We can show that in 6-dimensions

$$\int d\Omega = \pi^3 \tag{3.72}$$

Let

$$A'^2 = \frac{4ac - b^2}{4c} \tag{3.73}$$

Using Eqs. (3.70), (3.71), (3.72) and (3.73) in Eq. (3.69), we get

$$M_{12}^{I}(AB) = \bar{N}^2 t_1^0 t_2^0 (-4\mu d) \int_0^1\!\!\int_0^1 \frac{u\,du\,dv}{c^{3/2}} \int_0^\infty\!\!\int \frac{R^5 dR\, d\Omega}{(R^2 + A'^2)^3}$$

or

$$M_{12}^{I}(AB) = \bar{N}^2 t_1^0 t_2^0 (-4\mu d \pi^3) \int_0^1\!\!\int_0^1 \frac{u\,du\,dv}{c^{3/2}} \int_0^\infty \frac{R^5 dR}{(R^2 + A'^2)^3} \tag{3.74}$$

Now

$$\begin{aligned} \frac{R^5}{(R^2 + A'^2)^3} &= \frac{R}{R^2 + A'^2} - \frac{2RA'^2}{(R^2 + A'^2)^2} + \frac{RA'^4}{(R^2 + A'^2)^3} \\ &= \frac{R}{R^2 + A'^2} + \frac{d}{dR}\left[\frac{A'^2}{R^2 + A'^2} - \frac{1}{4}\frac{A'^4}{(R^2 + A'^2)^2}\right] \end{aligned}$$

$$\begin{aligned} \therefore \int_0^\infty \frac{R^5 dR}{(R^2 + A'^2)^3} &= \int_0^\infty \frac{R\,dR}{R^2 + A'^2} - \frac{3}{4} \qquad \mathrm{Re}\, A'^2 > 0 \\ &= \frac{1}{2} \log_e |R^2 + A'^2| \Big|_0^\infty - \frac{3}{4} \\ &= \left(\underset{R\to\infty}{\mathrm{Lt}} \tfrac{1}{2} \log_e R^2 - \tfrac{3}{4}\right) - \tfrac{1}{2} \log_e |A'^2| \qquad \mathrm{Re}\, A'^2 > 0 \\ &= \mathrm{Div}\, R - \tfrac{3}{4} - \tfrac{1}{2} \log |A'^2| \end{aligned}$$

where $\mathrm{Div}\, R = \underset{R\to\infty}{\mathrm{Lt}} (\log_e R)$

Since ${}^{(1)}M_{12}^{I}$ is given by Eq. (3.62), the factor $\mathrm{Div}\,R - 3/4 = \mathop{Lt}_{R\to\infty}\left(\frac{1}{2}\log_e R^2\right) - 3/4$ occurs twice with positive and negative signs. Therefore it does not contribute.

We shall prove for $\mathrm{Re}\,A'^2 < 0$ and $\mathrm{Im}\,A'^2 < 0$:

$$\int_0^\infty \frac{R\,dR}{(R^2 + A'^2)} = -\frac{1}{2}\log|A'^2| + i\,\pi/2 + \mathrm{Div}\,R$$

Let

$$A'^2 = -\left(A^2 + i\epsilon(1-u)\right)$$

$$\therefore \int_0^\infty \frac{R\,dR}{R^2 - (A^2 + i\epsilon(1-u))} = \int_0^\infty \frac{R\,dR}{\left(R - A - \frac{i\epsilon}{2}(1-u)\right)\left(R + A + \frac{i\epsilon}{2}(1-u)\right)}$$

$$= \frac{1}{2}\int_0^\infty dR\left[\frac{1}{R - A - \frac{i\epsilon}{2}(1-u)} + \frac{1}{R + A + \frac{i\epsilon}{2}(1-u)}\right]$$

$$= \frac{1}{2}\int_0^\infty dR\left[\frac{R - A + i\epsilon/2(1-u)}{(R-A)^2 + \left(\frac{\epsilon}{2}(1-u)\right)^2} + \frac{R + A - i\epsilon/2(1-u)}{(R+A)^2 + \left(\frac{\epsilon}{2}(1-u)\right)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log_e\left|(R-A)^2 + \epsilon^2/4(1-u)^2\right| + i\tan^{-1}\frac{R-A}{\epsilon/2(1-u)}\right.$$

$$\left.\left. + \frac{1}{2}\log_e\left|(R+A)^2 + \epsilon^2/4(1-u)^2\right| - i\tan^{-1}\frac{R+A}{\epsilon/2(1-u)}\right]\right|_0^\infty$$

$$= \frac{1}{2}\left[\mathop{Lt}_{R\to\infty}\left(\log_e R^2\right) - \log\left|A^2 + \epsilon^2/4(1-u)^2\right| + \right.$$

$$+ i\left(\pi/_2 - \tan^{-1}\frac{-A}{\epsilon/_2(1-u)}\right) - i\left(\frac{\pi}{2} - \tan^{-1}\frac{A}{\epsilon/_2(1-u)}\right)\Bigg]$$

$$= \frac{1}{2}\left[\underset{R\to\infty}{\mathrm{Lt}}\left(\log_e R^2\right) - \log_e\left|A^2 + \epsilon^2/_4(1-u)^2\right| + 2i\tan^{-1}\frac{A}{\epsilon/_2(1-u)}\right]$$

$$= \tfrac{1}{2}\left[\underset{R\to\infty}{\mathrm{Lt}}\left(\log_e R^2\right) - \log_e|A'^2| + 2i\cdot\pi/_2\right] \quad , \quad \epsilon \to 0$$

$$= -\tfrac{1}{2}\log_e|A'^2| + i\pi/_2 + \mathrm{Div}\ R$$

$$= -\tfrac{1}{2}\log_e|A'^2| + i\pi/_2$$

Omitting $\underset{R\to\infty}{\mathrm{Lt}}\left(\tfrac{1}{2}\log R^2\right)$ as it will cancel with other terms as discussed above.

We can write

$$\int_0^\infty \frac{R^5\,dR}{(R^2+A'^2)^3} = -\frac{1}{2}\left[\log_e|A'^2| - i\pi\,\theta(-A'^2) + \frac{3}{2}\right] \tag{3.75}$$

where

$$\left.\begin{aligned} \theta(-x) &= 1 \quad \text{if} \quad x < 0 \\ &= 0 \quad \text{if} \quad x > 0 \end{aligned}\right] \tag{3.76}$$

From Eqs. (3.74) and (3.75), we get

$$M_{12}^{I}(AB) = \left(\bar{N}^2 t_1^{\circ} t_2^{\circ} \cdot 2\mu_d\,\pi^3\right)\int_0^1\int_u^1 \frac{u\,du\,dv}{C^{3/2}}\left[\log_e|A'^2| - i\pi\,\theta(-A'^2) + 3/_2\right]$$

Using Eq. (3.62), we get

$$
\begin{aligned}
{}^{(1)}M_{12}^{I} &= M_{12}^{I}(\alpha\alpha) + M_{12}^{I}(\beta\beta) - M_{12}^{I}(\alpha\beta) - M_{12}^{I}(\beta\alpha) \\
&= \left(\bar{N}^2 t_1^0 t_2^0 \cdot 2\mu_d \pi^3\right) \int_0^1\!\!\int_0^1 \frac{u\,du\,dv}{C^{3/2}} \Bigg[\log_e \left| \frac{A'^2(\alpha\alpha)\cdot A'^2(\beta\beta)}{A'^2(\alpha\beta)\cdot A'^2(\beta\alpha)} \right| \\
&\qquad - i\pi\,\Theta(-A'^2(\alpha\alpha)) - i\pi\,\Theta(-A'^2(\beta\beta)) \\
&\qquad + i\pi\,\Theta(-A'^2(\alpha\beta)) + i\pi\,\Theta(-A'^2(\beta\alpha)) \Bigg]
\end{aligned}
\tag{3.77}
$$

where

$$
\begin{aligned}
A'^2 &= \frac{4ac - b^2}{4c} \\
&= X + Y \cos(\underline{q}, \underline{q}')
\end{aligned}
\tag{3.78}
$$

where

$$
\begin{aligned}
X = {} & \left[-\tfrac{u^2}{4}(2v^2 - 2v + 1) + \tfrac{u}{4} - \tfrac{\mu_d}{\mu_{Kd}}(1-u) \right] q^2 + \Big[u^4 \Big\{ (v^2 - \tfrac{v}{2})^2 \\
& + (\tfrac{1}{2} - \tfrac{3}{2}v + v^2)^2 \Big\} + u^2 (\tfrac{v^2}{2} - v + \tfrac{1}{4}) + 2u^3 \times \\
& \Big\{ -\tfrac{v}{2}(v^2 - \tfrac{v}{2}) - (1-v)(v - \tfrac{1}{2})(\tfrac{v-1}{2}) \Big\} \Big] \cdot \tfrac{q^2}{4c} + \\
& \left(A^2 v + B^2(1-v)\right) u + \alpha^2 (1-u),
\end{aligned}
\tag{3.79}
$$

$$Y = \frac{u^2}{2} v(1-v) q^2 - \left[2u^4 (v^2 - v/2)(1/2 - 3/2\, v + v^2) + \frac{u^2}{2} \cdot v(1-v) \right] \times q^2/4C \quad , \tag{3.80}$$

$$C = -u^2 (v - 1/2)^2 + u/4 \left(1 - 4 \frac{\mu_d}{\mu}\right) + \mu_d/\mu_{\pi d} , \tag{3.81}$$

and

$$I = \int_0^1 \int_0^1 \frac{u\, du\, dv}{C^{3/2}} \left[\log \left| \frac{A'^2(\alpha\alpha) \cdot A'^2(\beta\beta)}{A'^2(\alpha\beta) \cdot A'^2(\beta\alpha)} \right| - i\pi\, \theta(-A'^2(\alpha\alpha)) - i\pi \cdot \theta(-A'^2(\beta\beta)) + i\pi\, \theta(-A'^2(\alpha\beta)) + i\pi\, \theta(-A'^2(\beta\alpha)) \right] \tag{3.82}$$

2. **Computation of** $^{(2)}M_{12}^{I}$ **and** $^{(3)}M_{12}^{I}$. Having calculated $^{(1)}M_{12}^{I}$, we shall calculate $^{(2)}M_{12}^{I}$ from Eq. (3.55)

Using Eq. (3.52), we get

$$\langle q', \varphi | t_1 | \ell, \chi_t \rangle = \iint d\mathbf{r}\, d\rho\; e^{-i q'\cdot r} f^*(\rho)\, t_1\, e^{i \ell \cdot r} \cdot f(t) \cdot \frac{e^{itp} - e^{-\beta\rho}}{\rho}$$

Using Eq. (3.1), we get

$$\langle q', \varphi | t_1 | \ell, \chi \rangle = \iint d\mathbf{r}\, d\rho\; e^{-i q'\cdot r} \varphi^*(\rho)\, t_1^\circ\, \delta(r - \rho/2)\, e^{i \ell \cdot r} f(t) \frac{e^{it\rho} - e^{-\beta\rho}}{\rho}$$

$$= t_1^\circ \int d\rho\; e^{i(\ell - q')\cdot \rho/2}\, \varphi^*(\rho)\, f(t)\, \frac{e^{it\rho} - e^{-\beta\rho}}{\rho}$$

Substituting for $\varphi^*(\rho)$, we have

$$\langle q', \varphi | t_1 | \ell, \chi \rangle = (N t_1^\circ \cdot 2\pi) \int_0^\infty d\rho \int_{-1}^{1} dx\; e^{i(\ell - q')\cdot \rho/2} \left(e^{-\alpha\rho} - e^{-\beta\rho}\right) f(t) \times$$

$$\times \left(e^{it\rho} - e^{-\beta\rho}\right).$$

$$= (N t_1^\circ\, 2\pi) \int_0^\infty d\rho \left(\frac{e^{i|\ell - q'|\rho/2} - e^{-i|\ell - q'|\rho/2}}{i(\ell - q')\,\rho/2}\right) \left(e^{-\alpha\rho} - e^{-\beta\rho}\right)$$

$$\times f(t) \left(e^{it\rho} - e^{-\beta\rho}\right)$$

Introduce $\int_0^\infty e^{-y\rho}\,dy = 1/\rho$.

We get

$$\langle \underline{q}', \varphi | t_1 | \underline{\ell}, \chi\rangle = \frac{(N t_1^0 2\pi)}{\frac{1}{2}|\underline{\ell}-\underline{q}'|}\int_0^\infty dy \int_0^\infty d\rho \Big[e^{-(\alpha+y)\rho + i(t + |\underline{\ell}-\underline{q}'|)\rho/2} -$$

$$e^{-(\beta+y)\rho + i(t + \frac{1}{2}|\underline{\ell}-\underline{q}'|)\rho} - e^{-(\alpha+\beta+y)\rho + i|\underline{\ell}-\underline{q}'|\rho/2}$$

$$+ e^{-(2\beta+y)\rho + i|\underline{\ell}-\underline{q}'|\rho/2} - \left(\text{terms with } |\underline{\ell}-\underline{q}'| \to -|\underline{\ell}-\underline{q}'|\right)\Big] \times f(t)$$

$$= \frac{(N t_1^0 2\pi)}{\frac{1}{2}|\underline{\ell}-\underline{q}'|}\int_0^\infty dy \Big[\left(\alpha + y - i(t + \tfrac{1}{2}|\underline{\ell}-\underline{q}'|)\right)^{-1} - \left(\alpha + y - i(t - |\tfrac{\underline{\ell}-\underline{q}'}{2}|)\right)^{-1}$$

$$- \left(\beta + y - i(t + |\tfrac{\underline{\ell}-\underline{q}'}{2}|)\right)^{-1} + \left(\beta + y - i(t - |\tfrac{\underline{\ell}-\underline{q}'}{2}|)\right)^{-1}$$

$$+ \left(\alpha + \beta + y + i(\tfrac{\underline{\ell}-\underline{q}'}{2})\right)^{-1} - \left(\alpha + \beta + y - i|\tfrac{\underline{\ell}-\underline{q}'}{2}|\right)^{-1}$$

$$+ \left(2\beta + y - i|\tfrac{\underline{\ell}-\underline{q}'}{2}|\right)^{-1} - \left(2\beta + y + i|\tfrac{\underline{\ell}-\underline{q}'}{2}|\right)^{-1}\Big] f(t)$$

$$= (N t_1^0 \cdot 4\pi \cdot f(t)) \int_0^\infty dy \Big[\left((\alpha + y - it)^2 + (\tfrac{\underline{\ell}-\underline{q}'}{2})^2\right)^{-1} -$$

$$\left(\beta + y - it + (\tfrac{\underline{\ell}-\underline{q}'}{2})^2\right)^{-1} - \left((\alpha + \beta + y)^2 + (\tfrac{\underline{\ell}-\underline{q}'}{2})^2\right)^{-1}$$

$$+ \left((2\beta + y)^2 + (\tfrac{\underline{\ell}-\underline{q}'}{2})^2\right)^{-1}\Big] \tag{3.83}$$

We have already proved that

$$\langle \underline{\ell}, \underline{t} | t_2 | \underline{p}, \underline{q} \rangle = (2\pi)^3 \bar{N} t_2^\circ \left(\frac{1}{p^2+\alpha^2} - \frac{1}{p^2+\beta^2} \right) \tag{3.60}$$

$$\underline{p} = \underline{t} - \tfrac{1}{2}(\underline{\ell} - \underline{q}) \tag{3.60'}$$

Forward Scattering, $\underline{q} = \underline{q}'$

Using Eqs. (3.83, (3.60) in Eq. (3.55), we get

$$
\begin{aligned}
{}^{(2)}M_{12}^{I} = \frac{4\pi N \bar{N} t_1^\circ t_1^\circ (-2\mu_d)}{(2\pi)^3} \iint d\underline{\ell}\, d\underline{t} \int_0^\infty dy \Big\{ & \Big(\big((\alpha+\gamma-it)^2 + \big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\big)^{-1} - \\
& \big((\beta+\gamma-it)^2 + \big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\big)^{-1} - \big((\alpha+\beta+\gamma)^2 + \big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\big)^{-1} + \\
& \big((2\beta+\gamma)^2 + \big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\big)^{-1}\Big] \cdot \Big[\big(\big(\underline{t} - \tfrac{\underline{\ell}-\underline{q}}{2}\big)^2 + \alpha^2\big)^{-1} - \big(\big(\underline{t} - \tfrac{\underline{\ell}-\underline{q}}{2}\big)^2 + \beta^2\big)^{-1}\Big] \times \\
& \times \big(t^2 + \mu_d/\mu_{\pi d}(\ell^2 - q^2) + \alpha^2 - i\epsilon\big)^{-1} \times f(t) \Big\}
\end{aligned}
$$

Note: $i\epsilon$ has — sign before it.

Using

$$\frac{1}{a - i\epsilon} = P(1/a) + i\pi\delta(a)$$

we can write

$$ {}^{(2)}M_{12}^{I} = \mathrm{Re}\, {}^{(2)}M_{12}^{I} + \mathrm{Im}\, {}^{(2)}M_{12}^{I} ,$$

where

$$\mathrm{Re}\,{}^{(2)}M_{12}^{I} = \frac{4\pi N\bar{N}\, t_1^0 t_2^0 (-2\mu_d)}{(2\pi)^3} \iint d\underline{l}\, d\underline{t} \int_0^\infty dy \Bigg\{ \Bigg[\left((\alpha + y - it)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1} -$$

$$\left((\beta + y - it)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\alpha+\beta+y)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1} +$$

$$\left((2\beta+y)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1} \Bigg] \left[\left(\left(\underline{t} - \frac{\underline{l}-\underline{q}}{2}\right)^2 + \alpha^2 \right)^{-1} - \left(\left(\underline{t} - \frac{\underline{l}-\underline{q}}{2}\right)^2 + \beta^2 \right) \right]$$

$$\times \left(t^2 + \frac{\mu_d}{\mu_{Kd}} (l^2 - q^2) + \alpha^2 \right)^{-1} \times f(t) \Bigg\} \qquad (3.84)$$

$$\mathrm{Im}\,{}^{(2)}M_{12}^{I} = i\pi \cdot \frac{4\pi N\bar{N}\, t_1^0 t_2^0 (-2\mu_d)}{(2\pi)^3} \iint d\underline{l}\, d\underline{t} \int_0^\infty dy \Bigg\{ \Bigg[\left((\alpha + y - it)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1}$$

$$- \left((\beta + y - it)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\alpha+\beta+y)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1}$$

$$+ \left((2\beta+y)^2 + \left(\frac{\underline{l}-\underline{q}}{2}\right)^2 \right)^{-1} \Bigg] \Bigg[\left(\left(\underline{t} - \frac{\underline{l}-\underline{q}}{2}\right)^2 + \alpha^2 \right)^{-1} -$$

$$\left(\left(\underline{t} - \frac{\underline{l}-\underline{q}}{2}\right)^2 + \beta^2 \right)^{-1} \Bigg] \times f(t) \times \delta\left(t^2 + \frac{\mu_d}{\mu_{Kd}} (l^2 - q^2) + \alpha^2 \right) \Bigg\} \qquad (3.85)$$

From Eq. (3.56), we have

$$ {}^{(3)}M_{12}^{I} = Re\,{}^{(3)}M_{12}^{I} + Im\,{}^{(3)}M_{12}^{I} $$

$$ Re\,{}^{(3)}M_{12}^{I} = \frac{4\pi N\bar{N} t_1^\circ t_2^\circ (-2\mu_d)}{(2\pi)^3} \iint d\underline{l}\, d\underline{t} \int_0^\infty dy \Big\{ \Big[\Big((\alpha+y+\iota t)^2 + \big(\tfrac{\underline{l}-\underline{q}}{2}\big)^2\Big)^{-1} - \Big((\alpha+\beta+y)^2 + \big(\tfrac{\underline{l}-\underline{q}}{2}\big)^2\Big)^{-1} + \Big((2\beta+y)^2 + \big(\tfrac{\underline{l}-\underline{q}}{2}\big)^2\Big)^{-1} \Big] \times \Big[\Big(\big(\underline{t} + \tfrac{\underline{l}-\underline{q}}{2}\big)^2 + \alpha^2\Big)^{-1} - \Big(\big(\underline{t} + \tfrac{\underline{l}-\underline{q}}{2}\big)^2 + \beta^2\Big)^{-1} \Big] \times \Big(t^2 + \mu_d/\mu_{kd}(l^2-q^2) + \alpha^2\Big)^{-1} \times \overset{*}{f}(t) \Big\} $$

Comparing it with Eq. (3.84), we see that

$$ \frac{Re\,{}^{(3)}M_{12}^{I}}{t_1^\circ t_2^\circ} = \frac{\left(Re\,{}^{(2)}M_{12}^{I}\right)^*}{t_1^\circ t_2^\circ} \tag{3.84'} $$

Also

$$ Im\,{}^{(3)}M_{12}^{I} = \iota\pi \cdot \frac{4\pi N\bar{N} t_1^\circ t_2^\circ (-2\mu_d)}{(2\pi)^3} \iint d\underline{l}\, d\underline{t} \int_0^\infty dy \Big\{ \Big((\alpha+y+\iota t)^2 + \big(\tfrac{\underline{l}-\underline{q}}{2}\big)^2\Big)^{-1} - \Big((\beta+y+\iota t)^2 + \big(\tfrac{\underline{l}-\underline{q}}{2}\big)^2\Big)^{-1} - \Big((\alpha+\beta+y)^2 + \big(\tfrac{\underline{l}-\underline{q}}{2}\big)^2\Big)^{-1} + (2\beta+y)^2 + \big(\tfrac{\underline{l}-\underline{q}}{2}\big)^2 \Big] \Big[\Big(\big(\underline{t} + \tfrac{\underline{l}-\underline{q}}{2}\big)^2 + \alpha^2\Big)^{-1} - \Big(\big(\underline{t} + \tfrac{\underline{l}-\underline{q}}{2}\big)^2 + \beta^2\Big)^{-1} \Big] \times \overset{*}{f}(t) \times \delta\big(t^2 + \mu_d/\mu_{kd}(l^2-q^2) + \alpha^2\big) \Big\} $$

Comparing it with Eq. (3.85), we find that

$$ \frac{Im\,{}^{(3)}M_{12}^{I}}{\iota t_1^\circ t_2^\circ} = \frac{\left(Im\,{}^{2}M_{12}^{I}\right)^*}{\iota t_1^\circ t_2^\circ} \tag{3.85'} $$

It is obvious from Eq. (3.84) that if we make the transformation

$$\underline{\ell} \rightarrow \underline{\ell} + \underline{q} \tag{3.86}$$

we shall have the angular dependence in the integrand only in one of the denominators, and there is no angular integration with respect to $\underline{t}$. Eq. (3.84) becomes

$$\begin{aligned} R_e {}^{(2)}M_{n}^{I} = \frac{4\pi N\bar{N} t_i^{\circ} t_i^{\circ}(-2\mu\alpha)}{(2\pi)^3} \iint d\underline{\ell}\, d\underline{t} \int_0^{\infty} dy \Big\{ \Big[\big((\alpha+y-it)^2 + \ell^2/4\big)^{-1} - \\ \big((\beta+y-it)^2 + \ell^2/4\big)^{-1} + \big((\alpha+\beta+y)^2 + \ell^2/4\big)^{-1} + \big((2\beta+y)^2 + \ell^2/4\big)^{-1} \Big] \\ \times \Big[\big((\underline{t} - \underline{\ell}/2)^2 + \alpha^2\big)^{-1} - \big((\underline{t} - \underline{\ell}/2)^2 + \beta^2\big)^{-1} \Big] \times \\ \big(t^2 + \mu_d/\mu_{\Lambda d}(\ell^2 + 2\underline{\ell}\cdot\underline{q}) + \alpha^2\big)^{-1} \times f(t) \Big\} \end{aligned} \tag{3.87}$$

Further, we can simplify the integration over $\underline{t}$ by the following transformation

$$\left. \begin{aligned} &\ell = \ell' t \quad , \quad y = y' t \\ &\delta_1 = \alpha/t \quad ; \quad \delta_2 = \beta/t \quad ; \quad \delta_3 = (\alpha+\beta)/t \quad , \quad \delta_4 = 2\beta/t \end{aligned} \right] \tag{3.88}$$

Eq. (3.87) becomes

$$R_e {}^{(2)}M_{n}^{I} = \frac{4\pi N\bar{N} t_i^{\circ} t_i^{\circ}(-2\mu\alpha)}{(2\pi)^3} \iint \frac{d\underline{\ell}'\, d\underline{t}}{t^2} \int_0^{\infty} dy' \Big\{ \Big[\big((\delta_1 + y' - i)^2 + \ell'^2/4\big)^{-1} -$$

$$- \left((\delta_2 + y' - i)^2 + \ell'^2/4\right)^{-1} - \left((\delta_3 + y')^2 + \ell'^2/4\right)^{-1} + \left((\delta_4 + y')^2 + \ell'^2/4\right)^{-1}\Big]$$

$$\times \left[\left(1 + \ell'^2/4 - \ell' x_t + \delta_1^2\right)^{-1} - \left(1 + \ell'^2/4 - \ell' x_t + \delta_2^2\right)^{-1} \right] \times$$

$$\left(1 + \delta_1^2 + \mu_d/\mu_{\kappa d}\left(\ell'^2 + 2\ell' q/t\, x_\ell\right)\right)^{-1} \times f(t) \Big\}$$

In the evaluation of $Re\, {}^{(2)}M_{12}^{I}$, we need the following results

$$\int_0^\infty \frac{1\, dy}{(\delta + y - i)^2 + \ell^2/4} = \frac{i}{2\ell} \log_e \left| \frac{(1 + \ell/2)^2 + \delta^2}{(1 - \ell/2)^2 + \delta^2} \right| + \frac{T(\delta, \ell)}{\ell} \tag{3.89}$$

where

$$T(\delta, \ell) = \tan^{-1} \left| \delta/(1 - \ell/2) \right| - \tan^{-1} \left| \delta/(1 + \ell/2) \right| , \quad \ell \leqslant 2 \tag{3.90a}$$

$$= \pi - \tan^{-1} \left| \delta/(\ell/2 - 1) \right| - \tan^{-1} \left| \delta/(1 + \ell/2) \right| , \quad \ell > 2 \tag{3.90b}$$

$$\int_0^\infty \frac{dy}{(\delta + y)^2 + \ell^2/4} = 2/\ell \left[\pi/2 - \tan^{-1} 2\delta/\ell \right] \tag{3.91}$$

$$\int \frac{d\Omega}{1 + \ell^2/4 \pm \ell x + \delta^2} = \frac{2\pi}{\ell} \log_e \left| \frac{(1 + \ell/2)^2 + \delta^2}{(1 - \ell/2)^2 + \delta^2} \right| \tag{3.92}$$

$$\int \frac{d\Omega}{1+\delta^2+\mu_d/\mu(\ell^2 \pm 2\ell q/_t x)} = \pi \frac{\mu}{\mu_d} \cdot t/_{q\ell} \log_e \left| \frac{1+\delta_1^2+\mu_d/\mu_{Kd}(\ell^2+2\ell q/_t)}{1+\delta_1^2+\mu_d/\mu_{Kd}(\ell^2-2\ell q/_t)} \right| \qquad (3.93)$$

See Appendix (1) for Eq. (3.89)

To evaluate $Re\,^{(2)}M_{12}^{I}$, first perform the integration over y', using Eqs. (3.89) and (3.91), we get

$$Re\,^{(2)}M_{12}^{I} = \frac{4\pi N\bar{N}\, t_1^0 t_2^0 (-2\mu_d)}{(\;)^3} \iint d\ell' dt \left\{ \left[\frac{i}{2\ell'} \log_e \left| \frac{(1+\ell'/2)^2+\delta_1^2}{(1-\ell'/2)^2+\delta_1^2} \right| + \frac{T(\delta_1,\ell')}{\ell'} \right.\right.$$

$$- \frac{i}{2\ell'} \log_e \left| \frac{(1+\ell'/2)^2+\delta_2^2}{(1-\ell'/2)^2+\delta_2^2} \right| - \frac{T(\delta_2,\ell')}{\ell'} - \frac{2}{\ell'}\left(\pi/2 - \tan^{-1}\frac{2\delta_3}{\ell'}\right)$$

$$\left. + \;2/\ell'\left(\pi/2 - \tan^{-1}\frac{2\delta_4}{\ell'}\right)\right] \times \left[\left(1+\ell'^2/4 - \ell' x_t + \delta_1^2\right)^{-1} - \right.$$

$$\left.\left. - \left(1+\ell'^2/4 - \ell' x_t + \delta_2^2\right)^{-1} \right] \times \left(1+\delta_1^2+\mu_d/\mu_{Kd}(\ell'^2+2\ell' q/_t x_\ell)\right)^{-1} \times f(t) \right\}$$

Let

$$T(\delta_1,\delta_2,\delta_3,\delta_4,\ell) = T(\delta_1,\ell) - T(\delta_2,\ell) + 2\left(\tan^{-1}\frac{2\delta_3}{\ell} - \tan^{-1}\frac{2\delta_4}{\ell}\right) \qquad (3.94)$$

We get

$$Re^{(2)}M_{12}^{I} = \frac{4\pi N\bar{N}t_1^{o} t_2^{o}(-2\mu_d)}{(2\pi)^3}\int_0^{\infty} \ell' d\ell' \int_0^{\infty} dt \left[\frac{t}{2} \log_e \left| \frac{(1+\ell'/_2)^2+\delta_1^2}{(1-\ell'/_2)^2+\delta_1^2} \cdot \frac{(1-\ell'/_2)^2+\delta_2^2}{(1+\ell'/_2)^2+\delta_2^2} \right| \right.$$

$$\left. + \; T(\delta_1,\delta_2,\delta_3,\delta_4,\ell') \right] \cdot \int d\Omega_t \left[(1+\ell'^2/_4 - \ell' x_t + \delta_1^2)^{-1} - \right.$$

$$\left. (1+\ell'^2/_4 - \ell' x_t + \delta_2^2)^{-1} \right] \times \int \frac{d\Omega_\ell}{1+\delta_1^2+\mu_d/\mu_{Kd}(\ell'^2+2\ell' q/_t x_\ell)} \times f(t)$$

Using Eqs. (3.92) and (3.93), we get

$$Re^{(2)}M_{12}^{I} = \frac{4\pi N\bar{N}t_1^{o} t_2^{o}(-2\mu_d)}{(2\pi)^3}\int_0^{\infty} \ell' d\ell' \int_0^{\infty} dt \left[\frac{t}{2} \log_e \left| \frac{((1+\ell'/_2)^2+\delta_1^2)((1-\ell'/_2)^2+\delta_2^2)}{((1-\ell'/_2)^2+\delta_1^2)((1+\ell'/_2)^2+\delta_2^2)} \right| \right.$$

$$\left. + \; T(\delta_1,\delta_2,\delta_3,\delta_4,\ell') \right] \cdot \frac{2\pi}{\ell'} \log \left| \frac{(1+\ell'/_2)^2+\delta_1^2}{(1-\ell'/_2)^2+\delta_1^2} \cdot \frac{(1-\ell'/_2)^2+\delta_2^2}{(1+\ell'/_2)^2+\delta_2^2} \right| \times$$

$$\times \quad \pi \cdot \frac{\mu_{Kd}}{\mu_d} \cdot \frac{t}{q\ell'} \log_e \left| \frac{1+\delta_1^2+\mu_d/\mu_{Kd}(\ell'^2+2\ell' q/_t)}{1+\delta_1^2+\mu_d/\mu_{Kd}(\ell'^2-2\ell' q/_t)} \right| \times f(t)$$

Simplifying and using Eq. (3.58$'$), we get

$$Re^{(2)}M_{12}^{I} = \left(-t_1^{o} t_2^{o} \frac{N^2}{\pi^2}\mu_{Kd}\right) \int_0^{\infty} t\,dt \int_0^{\infty} \frac{d\ell'}{\ell'} \left[\frac{t}{2} \log_e \left| \frac{(1+\ell'/_2)^2+\delta_1^2}{(1-\ell'/_2)^2+\delta_1^2} \; \frac{(1-\ell'/_2)^2+\delta_2^2}{(1+\ell'/_2)^2+\delta_2^2} \right| \right.$$

$$+ \; T(\delta_1, \delta_2, \delta_3, \delta_4, \ell')\Big] \log_e \left| \frac{(1+\ell'/2)^2+\delta_1^2}{(1-\ell'/2)^2+\delta_1^2} \cdot \frac{(1-\ell'/2)^2+\delta_2^2}{(1+\ell'/2)^2+\delta_2^2} \right| \times$$

$$\frac{1}{q} \cdot \log_e \left| \frac{1+\delta_1^2+ \mu_d/\mu_{Kd}(\ell'^2+2\ell' q/t)}{1+\delta_1^2+\mu_d/\mu_{Kd}(\ell'^2-2\ell' q/t)} \right| \times f(t) \; .$$

Define

$$L(\ell) = \left| \frac{(1+\ell/2)^2+\delta_1^2}{(1-\ell/2)^2+\delta_1^2} \times \frac{(1-\ell/2)^2+\delta_2^2}{(1+\ell/2)^2+\delta_2^2} \right| \tag{3.95}$$

$$g(t) = \int_0^\infty \frac{d\ell'}{\ell'} \left(\log_e L(\ell)\right)^2 \frac{1}{q} \log_e \left| \frac{1+\delta_1^2+\frac{\mu_d}{\mu_{Kd}}(\ell'^2+2\ell' q/t)}{1+\delta_1^2+\frac{\mu_d}{\mu_{Kd}}(\ell'^2-2\ell' q/t)} \right| \tag{3.96}$$

$$k(t) = \int_0^\infty \frac{d\ell'}{\ell'} \cdot T(\delta_1,\delta_2,\delta_3,\delta_4,\ell')\left(\log_e L\right) \cdot \frac{1}{q} \log \left| \frac{1+\delta_1^2+\frac{\mu_d}{\mu_{Kd}}(\ell'^2+2\ell' q/t)}{1+\delta_1^2+\frac{\mu_d}{\mu_{Kd}}(\ell'^2-2\ell' q/t)} \right| \tag{3.97}$$

$$G(t) = 1/2 \; g(t) + k(t) \tag{3.98}$$

Using Eqs. (3.95), (3.96), (3.97) and (3.98), we get

$$Re\,{}^{(2)}M_{12}^{I} = -t_1^{\circ} t_2^{\circ} \frac{N^2}{\pi^2}\mu_{Kd} \int_0^\infty dt \; t\, G(t) \cdot f(t) \tag{3.99}$$

From Eq. (3.84$'$), we have

$$Re\,{}^{(3)}M_{12}^{I} = -t_1^{\circ} t_2^{\circ} \frac{N^2}{\pi^2}\mu_{Kd} \int_0^\infty dt \cdot t \; G^*(t)\, f^*(t) \tag{3.99$'$}$$

From Eq. (3.53), we have

$$f(t) = \frac{1}{At^4 + at^2 - it - b}$$

where

$$A = \frac{1}{2\beta(\alpha+\beta)^2}$$

$$a = \frac{\beta}{(\alpha+\beta)^2} + \frac{1}{2\beta}$$

$$b = \beta/2 - \beta^3/2(\alpha+\beta)^2$$

We have,

$$\left.\begin{aligned} f(t) - f^*(t) &= \frac{2it}{(At^4 + at^2 - b)^2 + t^2} \\ f(t) + f^*(t) &= \frac{2(At^4 + at^2 - b)}{(At^4 + at^2 - b)^2 + t^2} \end{aligned}\right] \tag{3.100'}$$

Adding up Eqs. (3.99) and (3.99') and using Eq. (3.100'), we get

$$\mathrm{Re}\left({}^{(2)}M_{12}^{I} + {}^{(3)}M_{12}^{I}\right) = -t_1^\circ t_2^\circ \frac{N^2}{\pi^2} \cdot \mu_{Kd} \int_0^\infty dt \left[\frac{-t^2 g(t)}{(At^4 + at^2 - b)^2 + t^2} + \frac{2t(At^4 + at^2 - b)\,h(t)}{(At^4 + at^2 - b)^2 + t^2}\right] \tag{3.100}$$

We shall find now $\underset{q\to 0}{\mathrm{Lt}}\, \mathrm{Re}\left({}^{(3)}M_{12}^{I} + {}^{0)}M_{12}^{I}\right)$

Using $\underset{x\to 0}{\mathrm{Lt}} \log(1+x) = x$ for $x<1$, we get

$$\underset{q\to 0}{\mathrm{Lt}}\ \tfrac{1}{q} \log_e \left| \frac{1+\delta_1^2 + \mu_d/\mu_{\kappa d}(\ell'^2 + 2\ell' q/t)}{1+\delta_1^2 + \mu_d/\mu_{\kappa d}(\ell'^2 - 2\ell' q/t)} \right| = \underset{q\to 0}{\mathrm{Lt}} \left[\log_e \left| 1 + 2\mu_d/\mu \cdot q/t \cdot \frac{\ell'}{(1+\delta_1^2 + \frac{\mu_d}{\mu_{\kappa d}}\ell'^2)} \right| \right.$$

$$\left. - \log_e \left| 1 - 2\mu_d/\mu_{\kappa d}\, q/t \frac{\ell'}{(1+\delta_1^2 + \mu_d/\mu\, \ell'^2)} \right| \right] = 4\frac{\mu_d}{\mu} \frac{\ell'}{t(1+\delta_1^2 + \frac{\mu_d}{\mu_{\kappa d}}\ell'^2)} \qquad (3.101)$$

From Eqs. (3.96) and (3.101), we have

$$g(t) = \int_0^\infty \frac{d\ell'}{\ell'} \left(\log_e L(\ell')\right)^2 \cdot 4\frac{\mu_d}{\mu} \cdot \frac{\ell'}{t\left(1+\delta_1^2 + \mu_d/\mu_{\kappa d}\,\ell'^2\right)}$$

$$= 4\frac{\mu_d}{\mu_{\kappa d}} \cdot \frac{1}{t} \int_0^\infty d\ell' \left(\log_e L(\ell')\right)^2 \cdot \frac{1}{\left(1+\delta_1^2 + \mu_d/\mu_{\kappa d}\,\ell'^2\right)}$$

$$= 4\frac{\mu_d}{\mu_{\kappa d}} \cdot \frac{1}{t} \cdot g_1(t),$$

where

$$g_1(t) = \int_0^\infty d\ell' \left(\log_e L(\ell')\right)^2 \frac{1}{\left(1+\delta_1^2 + \mu_d/\mu\, \ell'^2\right)} \qquad (3.102)$$

From Eqs. (3.97) and (3.101), we have

$$h(t) = \int_0^\infty \frac{d\ell'}{\ell'}\ T(\delta_1,\delta_2,\delta_3,\delta_4,\ell') \left(\log_e L(\ell')\right) \cdot 4\frac{\mu_d}{\mu} \cdot \frac{1}{t} \frac{\ell'}{(1+\delta_1^2 + \frac{\mu_d}{\mu_{\kappa d}}\ell'^2)}$$

$$= 4\frac{\mu_d}{\mu_{\kappa d}} \cdot \frac{1}{t} \int_0^\infty d\ell'\ T(\delta_1,\delta_2,\delta_3,\delta_4,\ell') \left(\log_e L(\ell')\right) \frac{1}{\left(1+\delta_1^2 + \mu_d/\mu_{\kappa d}\,\ell'^2\right)}$$

$$= 4\frac{\mu_d}{\mu_{\kappa d}} \cdot \frac{1}{t}\, h_1(t)$$

where

$$h_1(t) = \int_0^\infty dl' \cdot T(\delta_1, \delta_2, \delta_3, \delta_4, l')(\log_e L(l')) \cdot \frac{1}{(1+\delta_1^2 + \mu_d/\mu_{\kappa d}\, l'^2)} \qquad (3.103)$$

For $q = 0$, substituting Eqs. (3.102) and (3.103) for Eq. (3.100)

$$Re\left({}^{(2)}M_{12}^{I} + {}^{(3)}M_{12}^{I}\right) = -4\frac{N^2}{\pi^2} t_1^0 t_2^0 \mu_d \cdot \int_0^\infty dt \left[\frac{-t\, g_1(t) + 2(At^4 + at^2 - b)\, h_1(t)}{(At^4 + at^2 - b)^2 + t^2}\right]$$

$$= -4 z I \qquad (3.104)$$

where $$z = N^2/\pi^2 \cdot t_1^0 t_2^0 \mu_d \qquad (3.105)$$

and $$I = \int_0^\infty dt \left[\frac{-t\, g_1(t) + 2(At^4 + at^2 - b)\, h_1(t)}{(At^4 + at^2 - b)^2 + t^2}\right] \qquad (3.106)$$

Imaginary part of $\left({}^{(2)}M_{12}^{I} + {}^{(3)}M_{12}^{I}\right)$

Referring to Eq. (3.84)

$$\delta\left(t^2 + \mu_d/\mu\,(\ell^2 - q^2) + \alpha^2\right) = \frac{\delta\left(t - \sqrt{\mu_d/\mu_{\kappa d}(q^2 - \ell^2) - \alpha^2}\right) + \delta\left(t + \mu_d/\mu_{\kappa d}(q^2 - \ell^2) - \alpha^2\right)}{\left|2\sqrt{\mu_d/\mu_{\kappa d}(q^2 - \ell^2) - \alpha^2}\right|}$$

only $t = \sqrt{\mu_d/\mu_{\kappa d}(q^2 - \ell^2) - \alpha^2}$ contributes

Since t cannot be negative, maximum ℓ is given by

$$\frac{\mu_d}{\mu_{\kappa d}}(q^2 - \ell^2) - \alpha^2 = 0 \quad \text{or} \quad \ell = \sqrt{q^2 - \mu_{\kappa d}/\mu_d \cdot \alpha^2} = q_1$$

Therefore, ℓ varies from 0 to q_1 .

Let

$$\underline{L} = \frac{\underline{l}-\underline{q}}{2}$$

We get from Eq. (3.85)

$$\mathrm{Im}\,{}^{(2)}M_{12}^{I} = (i\pi)\cdot\frac{4\pi N\bar{N}\, t_1^\circ t_2^\circ (-2\mu_d)}{(2\pi)^3}\iint d\underline{l}\, d\underline{t}\int_0^\infty dy\left\{\left[\left((\alpha+y-it)^2+L^2\right)^{-1}-\right.\right.$$

$$\left. -\left((\beta+y-it)^2+L^2\right)^{-1}-\left((\alpha+\beta+y)^2+L^2\right)^{-1}+\left((2\beta+y)^2+L^2\right)^{-1}\right]\cdot$$

$$\left.\times\left[\left((\underline{t}-\underline{L})^2 \quad\right)^{-1}-\left((\underline{t}-\underline{L})^2+\beta^2\right)^{-1}\right]\times f(t)\times\delta\left(t^2+\frac{\mu_d}{\mu_{Kd}}(\ell^2-q^2)-\alpha^2\right)\right\}$$

Using Eq. (3.58$'$) and the limits of ℓ , we have

$$\mathrm{Im}\,{}^{(2)}M_{12}^{I} = -i\frac{N^2}{(2\pi)^3}\,2\mu_d\; t_1^\circ t_2^\circ\int_0^{q_1}\ell^2 d\ell\int d\Omega_\ell\int d\Omega_t\int_0^\infty dy\left\{\left[\left((\alpha+y-it)^2+L^2\right)^{-1}-\right.\right.$$

$$\left.-\left((\beta+y-it)^2+L^2\right)^{-1}-\left((\alpha+\beta+y)^2+L^2\right)^{-1}+\left((2\beta+y)^2+L^2\right)^{-1}\right]\times$$

$$\left.\times\left[\left((\underline{t}-\underline{L})^2+\alpha^2\right)^{-1}-\left((\underline{t}-\underline{L})^2+\beta^2\right)^{-1}\right]\cdot f(t)\cdot t\right\}_{t=\sqrt{\frac{\mu_d}{\mu_{Kd}}(q^2-\ell^2)-\alpha^2}}$$

In the above expression $t = \sqrt{\mu_d/\mu_{Kd}(q^2-\ell^2)-\alpha^2}$

Using Eq. (3.92), we have

$$\mathrm{Im}\,{}^{(2)}M_{12}^{I} = -i\frac{N^2}{(2\pi)^3}\cdot 2\mu_d\cdot t_1^0 t_2^0\int_0^{q_1} \ell^2 d\ell \int d\Omega_\ell \int_0^\infty dy \Big\{ \Big[\left((\alpha+y-it)^2+L^2\right)^{-1} -$$

$$\left((\beta+y-it)^2+L^2\right)^{-1} - \left((\alpha+\beta+y)^2+L^2\right)^{-1} + \left((2\beta+y)^2+L^2\right)^{-1} \Big] \times$$

$$\times \frac{\pi}{t}\cdot\frac{1}{L}\log_e\left|\frac{(t+L)^2+\alpha^2}{(t-L)^2+\alpha^2}\;\frac{(t-L)^2+\beta^2}{(t+L)^2+\beta^2}\right|\cdot t\cdot f(t)\Big\}_{t=\sqrt{\frac{\mu_d}{\mu_{Kd}}(q_1^2-\ell^2)-\alpha^2}}$$

$$= -i\frac{N^2}{(2\pi)^2}\mu_d\cdot t_1^0 t_2^0\int_0^{q_1} \ell^2 d\ell \int d\Omega_\ell \int_0^\infty dy \Big\{ \Big[\left((\alpha+y-it)^2+L^2\right)^{-1} -$$

$$\left((\beta+y-it)^2+L^2\right)^{-1} - \left((\alpha+\beta+y)^2+L^2\right)^{-1} + \left((2\beta+y)^2+L^2\right)^{-1} \Big] \times$$

$$\times \frac{1}{L}\log_e\left|\frac{(t+L)^2+\alpha^2}{(t-L)^2+\alpha^2}\;\frac{(t-L)^2+\beta^2}{(t+L)^2+\beta^2}\right|\cdot f(t)\Big\}_{t=\sqrt{\frac{\mu_d}{\mu_{Kd}}(q_1^2-\ell^2)-\alpha^2}}$$

Now

$$\frac{1}{(y+\alpha-it)^2+L^2} = \frac{1}{2iL}\left[\frac{1}{y+\alpha-i(t+L)} - \frac{1}{y+\alpha-i(t-L)}\right]$$

We get

$$\int_0^\infty \frac{dy}{y+\alpha-i(t+L)} = \tfrac{1}{2}\log\left|(y+\alpha)^2+(t+L)^2\right|\Big|_0^\infty + i\left(\pi/2 - \tan^{-1}\left|\frac{\alpha}{t+L}\right|\right)$$

and

$$\int_0^\infty \frac{dy}{y+\alpha-i(t-L)} = \tfrac{1}{2}\log\left|(y+\alpha)^2+(t-L)^2\right|\Big|_0^\infty + \frac{i(t-L)}{|t-L|}\cdot\left(\pi/2 - \tan^{-1}\left|\alpha/t-L\right|\right)$$

Therefore

$$\int_0^\infty \frac{dy}{(y+\alpha-it)^2+L^2} = \frac{1}{2iL}\left[\frac{1}{2}\log\left|\frac{(y+\alpha)^2+(t+L)^2}{(y+\alpha)^2+(t-L)^2}\right| + i\left(\pi/2 - \tan^{-1}\left|\frac{\alpha}{t+L}\right|\right)\right.$$

$$\left. - \frac{i(t-L)}{|t-L|}\cdot\left(\pi/2 - \tan^{-1}\left|\alpha/t-L\right|\right)\right]$$

$$= 1/4L \log_e\left|\frac{(y+\alpha)^2+(t-L)^2}{(y+\alpha)^2+(t+L)^2}\right| + \frac{T(\alpha,L)}{2L} \qquad (3.107)$$

where

$$T(\alpha,L) = \left(\pi/2 - \tan^{-1}\left|\alpha/t+L\right|\right) - \frac{t-L}{|t-L|}\left(\pi/2 - \tan^{-1}\left|\alpha/t-L\right|\right) \qquad (3.108)$$

Using Eqs. (3.107) and (3.91), we get by integrating over y

$$\mathscr{I}m\, {}^{(2)}M_{12}^{I} = -i\frac{N^2}{(2\pi)^2}\mu_d\, t_1^0 t_2^0 \int_0^{q_1} \ell^2 d\ell \int d\Omega_\ell \left\{\left[\, 1/4L \log_e\left|\frac{\alpha^2+(t-L)^2}{\alpha^2+(t+L)^2}\right| + \frac{T(\alpha,L)}{2L}\right.\right.$$

$$- \left(1/4L \log_e\left|\frac{\beta^2+(t-L)^2}{\beta^2+(t+L)^2}\right| + \frac{T(\beta,L)}{2L}\right) - \frac{1}{L}\left(\pi/2 - \tan^{-1}\left|\frac{\alpha+\beta}{L}\right|\right)$$

$$\left.\left. + 1/L\cdot\left(\pi/2 - \tan^{-1}\left|2\beta/L\right|\right)\right]\frac{1}{L}\log\left|\frac{(t+L)^2+\alpha^2}{(t-L)^2+\alpha^2}\cdot\frac{(t-L)^2+\beta^2}{(t+L)^2+\beta^2}\right| \times f(t)\right\}_{t=\sqrt{M/\mu_{\kappa d}(q^2-\ell^2)-\alpha^2}}$$

$$= -i\frac{N^2}{(2\pi)^2}\mu_d\, t_1^\circ t_2^\circ \int_0^{q_1} \ell^2 d\ell \int \frac{d\Omega_\ell}{L^2}\Big\{\Big[-\tfrac{1}{4}(\log_e M) + \frac{T(\alpha,L)}{2} - \frac{T(\beta,L)}{2} + \tan^{-1}\left|\frac{\alpha+\beta}{L}\right| - \tan^{-1}\left|\frac{2\beta}{L}\right|\Big](\log_e M)\times f(t)\Big\}_{t=\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}}$$

where

$$M = \left|\frac{\alpha^2+(t+L)^2}{\alpha^2+(t-L)^2} \times \frac{\beta^2+(t-L)^2}{\beta^2+(t+L)^2}\right| \qquad (3.109)$$

Let

$$T = T(\alpha,L) - T(\beta,L) + 2\left(\tan^{-1}\left|\frac{\alpha+\beta}{L}\right| - \tan^{-1}\left|\frac{2\beta}{L}\right|\right) \qquad (3.110)$$

Using Eq. (3.110), we get

$$\mathrm{Im}\,{}^{(2)}M_{12}^{I} = -i\frac{N^2}{(2\pi)^2}\cdot\frac{\mu_d}{2}t_1^\circ t_2^\circ\int_0^{q_1}\ell^2 d\ell\int\frac{d\Omega_\ell}{L^2}\Big[-\tfrac{1}{2}\log M + T\Big](\log_e M)\times f(t)\Big|_{t=\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}}$$

Comparing Eqs. (3.85) and 3.85'), we find

$$\mathrm{Im}\,{}^{(3)}M_{12}^{I} = -i\frac{N^2}{(2\pi)^2}\frac{\mu_d}{2}t_1^\circ t_2^\circ\int_0^{q_1}\ell^2 d\ell\int\frac{d\Omega_\ell}{L^2}\Big[\tfrac{1}{2}\log M + T\Big](\log_e M)\times f^*(t)\Big|_{t=\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}}$$

We get the imaginary part from the last two expressions,

$$\mathrm{Im}\left({}^{(2)}M_{12}^{I} + {}^{(4)}M_{12}^{I}\right) = -i\frac{N^2}{(2\pi)^2}\cdot\frac{\mu_d}{2}t_1^\circ t_2^\circ\int_0^{q_1}\ell^2 d\ell\int\frac{d\Omega_\ell}{L^2}\left\{\left[\frac{t\log M + 2(At^4+at^2-b)T}{(At^4+at^2-b)^2+t^2}\right]\times(\log_e M)\right\}_{t=\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}} \qquad (3.111)$$

where, we have used the results of Eq. (3.100)

3. __Computation of__ $^{(4)}M_{12}^{I}$. We shall calculate $^{(4)}M_{12}^{I}$ now for $\underline{q} = \underline{q}'$. Referring to Eq. (3.57), we have

$$^{(4)}M_{12}^{I} = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^3} \cdot \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \chi_t \rangle \langle \chi_t, \underline{\ell} | t_2 | \varphi, \underline{q} \rangle \times \times \left(q^2/2\mu_{K\alpha} + \epsilon_B - E_\ell - E_t + i\epsilon \right)^{-1} \qquad (3.57)$$

Using Eq. (3.83), we get

$$^{(4)}M_{12}^{I} = \frac{(-2\mu_\alpha)(4\pi N)^2 t_1^0 t_2^0}{(2\pi)^6} \iint d\underline{t}\, d\underline{\ell} \int dy_1 \Big[\left((\alpha + y_1 - it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\beta + y_1 - it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\alpha+\beta+y_1)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} + \left((2\beta + y_1)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} \Big]$$
$$\times \int_0^\infty dy_2 \Big[\left((\alpha + y_2 + it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\beta + y_2 + it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\alpha+\beta+y_2)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} + \left((2\beta + y_2)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} \Big] \frac{f(t) f^*(t)}{\left(t^2 + \mu_\alpha/\mu_{K\alpha}(\ell^2 - q^2) + \alpha^2 - i\epsilon \right)}$$

$$= \mathrm{Re}\, ^{(4)}M_{12}^{I} + \mathrm{Im}\, ^{(4)}M_{12}^{I}$$

where,

$$\mathrm{Re}\, ^{(4)}M_{12}^{I} = \frac{-8\mu_\alpha N^2 t_1^0 t_2^0}{(2\pi)^4} \iint d\underline{\ell}\, d\underline{t} \int_0^\infty dy_1 \Big[\left((\alpha + y_1 - it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\beta + y_1 - it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\alpha+\beta+y_1)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} + \left((2\beta + y_1)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} \Big]$$
$$\times \int dy_2 \Big[\left((\alpha + y_2 + it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\beta + y_2 + it)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} - \left((\alpha+\beta+y_2)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} + \left((2\beta + y_2)^2 + \left(\tfrac{\underline{\ell}-\underline{q}}{2}\right)^2 \right)^{-1} \Big] \frac{f(t) \cdot f^*(t)}{\left(t^2 + \mu_\alpha/\mu_{K\alpha}(\ell^2 - q^2) + \alpha^2 \right)} \qquad (3.112)$$

and

$$\mathcal{I}m\,{}^{(4)}M_{12}^{I} = \frac{(i\pi)(-8\mu_d N^2 t_1^\circ t_2^\circ)}{(2\pi)^4}\iint d\underline{t}\, d\underline{\ell}\int_0^\infty dy_1 \Big[\Big((\alpha+y_1-it)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1} -$$

$$\Big((\beta+y_1-it)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1} - \Big((\alpha+\beta+y_1)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1} + \Big((2\beta+y_1)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1}\Big]$$

$$\times \int_0^\infty dy_2 \Big[\Big((\alpha+y_2+it)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1} - \Big((\beta+y_2+it)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1} -$$

$$\Big((\alpha+\beta+y_2)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1} + \Big((2\beta+y_2)^2+\big(\tfrac{\underline{\ell}-\underline{q}}{2}\big)^2\Big)^{-1}\Big] \; f(t)\,f^*(t)\;\times$$

$$\times \;\delta\big(t^2+\mu_d/\mu_{kd}(\ell^2-q^2)+\alpha^2\big) \qquad (3.113)$$

We shall compute now, $Re\,{}^{(4)}M_{12}^{I}$

As for $Re\,{}^{(2)}M_{12}^{I}$, make the transformation $\underline{\ell}\to\underline{\ell}+\underline{q}$ in Eq. (3.112) and let

$$\ell = \ell' t \;;\quad y_1 = y_1' t \;;\quad y_2 = y_2' t$$

$$\delta_1 = \alpha/t \;;\quad \delta_2 = \beta/t \;,\quad \delta_3 = (\alpha+\beta)/t \;;\quad \delta_4 = 2\beta/t \;.$$

in Eq. (3.112)

$$Re\,{}^{(4)}M_{12}^{I} = \frac{-8\mu_d N^2 t_1^\circ t_2^\circ}{(2\pi)^4}\int d\underline{\ell}'\int_0^\infty t\,dt\int_0^\infty dy_1'\Big[\big((\delta_1+y_1'-i)^2+\ell'^2/4\big)^{-1} - \big((\delta_2+y_1'-i)^2+\ell'^2/4\big)^{-1} -$$

$$\big((\delta_3+y_1')^2+\ell'^2/4\big)^{-1} + \big((\delta_4+y_1')^2+\ell'^2/4\big)^{-1}\Big]\times\int_0 dy_2'\Big[\big((\delta_1+y_2'+i)^2+\ell'^2/4\big)^{-1} -$$

$$\big((\delta_3+y_2')^2+\ell'^2/4\big)^{-1} + \big((\delta_4+y_2')^2+\ell'^2/4\big)^{-1}\Big]\times f(t)\,f^*(t)\times\int\frac{d\Omega_t}{1+\delta_1^2+\mu_d/\mu_{kd}(\ell'^2+2\ell'\frac{q}{t}x_t)}$$

Using Eqs. (3.89), (3.91) and (3.93), we get

$$
\begin{aligned}
\mathrm{Re}^{(4)} M_{12}^{I} = & -\frac{8\mu_d N^2 t_i^0 t_i^0}{(2\pi)^4} \int d\ell' \int_0^\infty t\, dt \Bigg\{ \Bigg[\frac{i}{2\ell'} \log_e \left| \frac{(1+\ell'/2)^2+\delta_1^2}{(1-\ell'/2)^2+\delta_1^2} \right| + \frac{T(\delta_1,\ell')}{\ell'} \\
& - \left(\frac{i}{2\ell'} \log \left| \frac{(1+\ell'/2)^2+\delta_2^2}{(1-\ell'/2)^2+\delta_2^2} \right| + \frac{T(\delta_2,\ell')}{\ell'} \right) - \frac{2}{\ell'} \left(\pi/2 - \tan^{-1} \frac{2\delta_3}{\ell'} \right) \\
& + \frac{2}{\ell'} \left(\pi/2 - \tan^{-1} \frac{2\delta_4}{\ell'} \right) \Bigg] \times \Bigg[-\frac{i}{2\ell'} \log_e \left| \frac{(1+\ell'/2)^2+\delta_1^2}{(1-\ell'/2)^2+\delta_1^2} \right| - \frac{T(\delta_1,\ell')}{\ell'} \\
& + i/2\ell' \log_e \left| \frac{(1+\ell'/2)^2+\delta_2^2}{(1-\ell'/2)^2+\delta_2^2} \right| - \frac{T(\delta_2,\ell')}{\ell'} - \frac{2}{\ell'} \left(\pi/2 - \tan^{-1} \frac{2\delta_3}{\ell'} \right) + \\
& + 2/\ell' \left(\pi/2 - \tan^{-1} \frac{2\delta_4}{\ell'} \right) \Bigg] \times \left(\frac{\pi \mu_{Kd} t}{\mu_d\, q \ell'} \right) \log \left| \frac{1+\delta_1^2 + \mu_d/\mu_{Kd}(\ell'^2 + 2\ell' q/t)}{1+\delta_1^2 + \mu_d/\mu_{Kd}(\ell'^2 - 2\ell' q/t)} \right| \times \\
& \times f(t) f^*(t) \Bigg\}
\end{aligned}
$$

Using Eq. (3.94), we get

$$
\begin{aligned}
\mathrm{Re}^{(4)} M_{12}^{I} = & -\frac{\mu_{Kd} N^2 t_i^0 t_i^0}{\pi^2} \int_0^\infty t^2 dt \int_0^\infty \frac{d\ell'}{\ell'} \Bigg\{ \Big[i/2 \left(\log_e L(\ell') \right) + T(\delta_1,\delta_2,\delta_3,\delta_4,\ell') \Big] \times \\
& \Big[-\frac{i}{2} \left(\log_e L(\ell') \right) + T(\delta_1,\delta_2,\delta_3,\delta_4,\ell') \Big] \times \\
& \frac{1}{q} \log \left| \frac{1+\delta_1^2 + \mu_d/\mu_{Kd}(\ell'^2 + 2\ell' q/t)}{1+\delta_1^2 + \mu_d/\mu_{Kd}(\ell'^2 - 2\ell' q/t)} \right| \times f(t) f^*(t) \Bigg\} \\
= & -\frac{\mu_{Kd} N^2 t_i^0 t_i^0}{\pi^2} \int_0^\infty t^2 dt \int_0^\infty \frac{d\ell'}{\ell'} \left[\frac{1}{4} \left(\log_e L(\ell') \right)^2 + \left(T(\delta_1,\delta_2,\delta_3,\delta_4,\ell') \right)^2 \right] \times \\
& \times \frac{1}{q} \log_e \left| \frac{1+\delta_1^2 + \mu_d/\mu_{Kd}(\ell'^2 + 2\ell' q/t)}{1+\delta_1^2 + \mu_d/\mu_{Kd}(\ell'^2 - 2\ell' q/t)} \right| \times \frac{1}{(At^4 + at^2 - b)^2 + t^2}
\end{aligned}
$$

Using Eq. (3.96), we get

$$\mathrm{Re}\,{}^{(4)}M_{12}^{I} = -\frac{\mu_{Kd}N^2 t_1^0 t_2^0}{\pi^2}\int_0^\infty dt\left[\frac{1}{4}\,g(t) + K(t)\right]\frac{t^2}{(At^4+at^2-b)^2+t^2} \qquad (3.114)$$

where

$$K(t) = \int_0^\infty \frac{d\ell'}{\ell'}\left(T(\delta_1,\delta_2,\delta_3,\delta_4,\ell')\right)^2 \frac{1}{q}\log\left|\frac{1+\delta_1^2+\frac{\mu_d}{\mu_{Kd}}(\ell'^2+2\ell' q/t)}{1+\delta_1^2+\mu_d/\mu_{Kd}(\ell'^2-2\ell' q/t)}\right| \qquad (3.115)$$

We shall compute now $\mathcal{I}m\,{}^{(4)}M_{12}^{I}$

For Eq. (3.113) using the calculations given in $\mathcal{I}m\,{}^{(2)}M_{12}^{I}$, we get

$$\mathcal{I}m\,{}^{(4)}M_{12}^{I} = -i\,\frac{\mu_d N^2 t_1^0 t_2^0}{2\pi^3}\iint dt\,d\ell\left\{\left[\frac{-i}{4L}\log M + \frac{T}{2L}\right]\times\right.$$

$$\left.\times\left[\frac{i}{4L}\log M + \frac{T}{2L}\right] f(t)f^*(t)\,\delta\!\left(t^2+\mu_d/\mu_{Kd}(\ell^2-q^2)+\alpha^2\right)\right\}$$

where M and T and defined by Eqs. (3. 109) and (3.110)

i.e. we have used the result

$$\int_0^\infty dy\left[\left((\alpha+\gamma_1-it)^2+L^2\right)^{-1} - \left((\beta+y-it)^2+L^2\right)^{-1} - \left((\alpha+\beta+y)^2+L^2\right)^{-1} +\right.$$

$$\left.\left((2\beta+\gamma)^2+L^2\right)^{-1}\right] = -i/4L\,\log M + T/2L$$

Using the properties of δ-function, we get

$$\mathrm{Im}\, {}^{(4)}M_{12}^{I} = -\frac{i\mu_d N^2 t_1^0 t_2^0}{(2\pi)^3} \iint \frac{d\underline{t}\, d\underline{\ell}}{L^2} \left[\frac{1}{4} (\log M)^2 + T^2 \right] \times f(t) f^*(t) \times$$

$$\cdot \frac{\delta\left(t - \sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}\right) + \delta\left(t + \sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}\right)}{\left|2\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}\right|}$$

$$= -\frac{i\mu_d N^2 t_1^0 t_2^0}{2(2\pi)^3} \int d\Omega_t \int \frac{d\underline{\ell}}{L^2} \left[\tfrac{1}{4} (\log M)^2 + T^2 \right] f(t) f^*(t)\, t \Big|_{t=\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}}$$

$$= -\frac{i\mu_d N^2 t_1^0 t_2^0}{(2\pi)^2} \int \frac{d\underline{\ell}}{L^2} \left[\tfrac{1}{4} (\log M)^2 + T^2 \right] f(t) f^*(t)\, t \Big|_{t=\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}}$$

ℓ varies from 0 to $q_1 = \sqrt{q^2 - \mu_d/\mu_{kd}\,\alpha^2}$, as before

$$\mathrm{Im}\, {}^{(4)}M_{12}^{I} = -\frac{i\mu_d N^2 t_1^0 t_2^0}{(2\pi)^2} \int_0^{q_1} \ell^2 d\ell \int \frac{d\Omega_\ell}{L^2} \left[\tfrac{1}{4} (\log M)^2 + T^2 \right] \times$$

$$\times \frac{t}{(At^4 + at^2 - b)^2 + t^2} \Bigg|_{t=\sqrt{\mu_d/\mu_{kd}(q^2-\ell^2)-\alpha^2}} \tag{3.116}$$

We shall find now $\underset{q\to 0}{Lt}\ Re^{(4)}M_{12}^{I}$.

Using Eq. (3.101) in Eq. (3.115), we find

$$K(t) = \int_0^\infty \frac{d\ell'}{\ell'} \frac{(T(\delta_1, \delta_2, \delta_3, \delta_4, \ell'))^2}{1+\delta_1^2 + \mu d/\mu\, \ell'^2} \cdot \frac{4\mu d}{\mu} \cdot \frac{1}{t}$$

$$= \frac{4\mu d}{\mu_{Kd}} \cdot \frac{1}{t} \cdot K_1(t)$$

where

$$K_1(t) = \int_0^\infty \frac{d\ell'}{\ell'} \cdot \frac{(T(\delta_1, \delta_2, \delta_3, \delta_4, \ell'))^2}{(1+\delta_1^2 + \mu d/\mu_{Kd}\, \ell'^2)} \qquad (3.117)$$

Using Eqs. (3.102) and (3.117), we get for $q = 0$

$$Re^{(4)}M_{12}^{I} = -4\mu d \frac{N^2}{\pi^2} t_1^0 t_2^0 \int_0^\infty dt \left(\frac{1}{4} g_1(t) + K_1(t)\right) \frac{t}{(At^4 + at^2 - \Delta)^2 + t^2} \qquad (3.118)$$

G. **Summary of the Calculations.**

M_{12}^{C} is given by Eq. (3.42)

For $q = 0$ M_{12}^{C} is given by Eq. (3.43)

$$M_{12}^{I} = {}^{(1)}M_{12}^{I} + {}^{(2)}M_{12}^{I} + {}^{(3)}M_{12}^{I} + {}^{(4)}M_{12}^{I} + (1 \leftrightarrow 2)$$

${}^{(1)}M_{12}^{I}$ is given by Eq. (3.77)

$\mathrm{Re}\left({}^{(2)}M_{12}^{I} + {}^{(3)}M_{12}^{I}\right)$ is given by Eq. (3.100)

$\mathrm{Im}\left({}^{(2)}M_{12}^{I} + {}^{(3)}M_{12}^{I}\right)$ is given by Eq. (3.111)

$\mathrm{Re}\,{}^{(4)}M_{12}^{I}$ is given by Eq. (3.114)

$\mathrm{Im}\,{}^{(4)}M_{12}^{I}$ is given by Eq. (3.116)

For $q = 0$

$\mathrm{Re}\left({}^{(2)}M_{12}^{I} + {}^{(4)}M_{12}^{I}\right)$ is given by Eq. (3.104)

$\mathrm{Re}\,{}^{(4)}M_{12}^{I}$ is given by Eq. (3.118)

The matrix element for the scattering of the K^--meson from the from the deuteron is given by

$$M = M_1 + M_2 + M_{12}^{C} + M_{12}^{I} \tag{2.22}$$

H. Correction to Elastic Scattering Due to Charge Exchange in the Intermediate State. We shall calculate now the contribution due to charge exchange

1. $K^- + p \rightarrow \bar{K}^0 + n$
2. $\bar{K}^0 + n \rightarrow K^- + p$

We shall suppose that

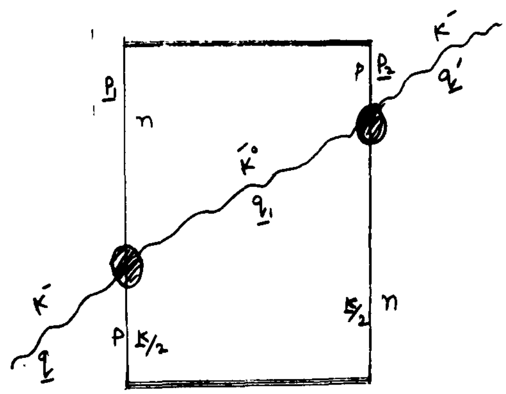
$$t_1^{ex} = t_1^{0\,ex}\,\delta(\mathbf{r} - \boldsymbol{\rho}/2) \qquad \text{for the process 1.} \qquad (3.119)$$

$$t_2^{ex} = t_2^{0\,ex}\,\delta(\mathbf{r} + \boldsymbol{\rho}/2) \qquad \text{for the process 2.} \qquad (3.120)$$

where $t_1^{0\,ex}$ and $t_2^{0\,ex}$ are constants and will be determined in Chapter VII. There is no direct charge exchange of the K^--meson with the neutron.

The Feynmann diagram for the charge exchange in the intermediate state is shown in Fig. 3.

$\underline{P}_1 = \underline{q} + \underline{K}/2 - \underline{q}_1$



$\underline{P}_2 = \underline{q}_1 + \underline{K}/2 - \underline{q}'$

Figure 3.

The matrix element for this process is given by

$$\mathcal{M}_{12}^{ex} = \langle \psi_f | t_2^{ex}\, G_{nn}\, t_1^{ex} | \psi_i \rangle \qquad (3.121)$$

$$= \langle \Psi_f | t_2^{ex} (E - h_{q_1} - h_{p_1} - h_{p_2} - U_{nn} + i\epsilon)^{-1} t_1^{ex} | \Psi_i \rangle \qquad (3.122)$$

Unn is the separable non-local potential between the two neutrons in the intermediate state.

Ψ_i and Ψ_f are defined in Eqs. (2.1) and (2.2).

Introducing

$$\sum_{q_1} |q_1\rangle\langle q_1| = \int \frac{dq_1}{(2\pi)^3} |q_1(r_0)\rangle\langle q_1(r_0)|$$

$$\sum_{p_1} |\psi_{p_1}\rangle\langle \psi_{p_1}| = \int dp_1 \, |\psi_{p_1}(r_1)\rangle\langle \psi_{p_2}(r_2)|$$

and similarly for Ψ_{r_2} in Eq. (3.122), we get

$$\mathcal{M}_{12}^{ex} = \iiint \langle \psi_f | t_2^{ex} | q_1, \psi_{p_1}(r_1); \psi_{p_2}(r_2) \rangle (E - E_{q_1} - E_{p_1} - E_{p_2} + i\epsilon)^{-1} \times$$

$$\times \langle \psi_{p_2}(r_2), \psi_{p_1}(r_1), q_1 | t_1^{ex} | \psi_i \rangle \frac{dq_1}{(2\pi)^3} dp_1 \, dp_2$$

In writing the above expression, we do not have to consider the Pauli exclusion principle, since the additional terms obtained thereby, are already included as part of the single scattering. Fig. 4 represents such a diagram.

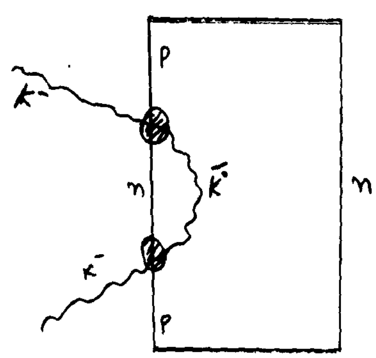


Figure 4.

This actually represents impulse approximation contribution. There is no double scattering taking place at all as the K^--meson is being scattered from only one particle.

Using Eqs. (2.3) and (2.4), we can write Eq. (3.121) in the center of mass system. The matrix element in the center of mass system is given by

$$M_{12}^{ex} = \langle \underline{q}', \varphi | t_2^{ex} \; G_{nn} t_1^{ex} | \varphi, \underline{q} \rangle \tag{3.123}$$

where $\underline{q}$ and $\underline{q}'$ are defined in center of mass system.

In the plane wave approximation, we have

$Gnn = G$ with $U = 0$

Proceeding exactly as in Chapter II, we can write

$$M_{12}^{ex} \simeq \left({}^{(1)}M_{12}^{I}\right)^{ex} \tag{3.124}$$

Comparing this expression with Eq. (2.25), we can write

$$M_{12}^{ex} \simeq \frac{t_1^{0ex} \cdot t_2^{0ex}}{2 t_1^0 t_2^0} \left({}^{(1)}M_{12}^{I} + 1 \leftrightarrow 2\right) \tag{3.125}$$

The factor of $\frac{1}{2}$ appears in the above expression because in Fig. 3, we can not exchange the proton and the neutron while the expressions for $\left({}^{(1)}M_{12}^{I} + (1 \leftrightarrow 2)\right)$ includes the exchange of the proton and the neutron as shown in Eq. (2.25).

We can define the total contribution by

$$M^T = M + M_{12}^{ex} \tag{3.126}$$

The treatment for charge exchange is not exact, but does give an order of magnitude of the correction to the elastic scattering cross sections.

CHAPTER IV

NUMERICAL EVALUATION

<u>Single Scattering</u>. For any q_L and $\theta = 0$

$$M_1 + M_2 = t_1^0 + t_2^0 \tag{3.25}$$

<u>Double Scattering</u>. The calculations have been done at three energies $q_L = 0$ Mev/c, 105 Mev/c and 194 Mev/c

<u>Bound State</u>. The results for the bound state forward scattering are given in Table I.

TABLE I
Values of the matrix element M_{12}^C

q_L (Mev/c)	q (f^{-1})	M_{12}^c / Z
0	0	- 27.4 - i 0
105	.42	- 16.4 - i 12.9
194	.78	- 10.1 - i 13.9

where $Z = \mu d/\pi^2 \cdot N^2 t_1^0 t_2^0$ (3.105)

Continuum State. The results for the continuum state forward scattering are given in Table II.

TABLE II
Values of the matrix element M^I_{12}

q_L (MeV/c)	${}^{(1)}M^I_{12}/Z$	$({}^{(2)}M^I_{12} + {}^{(3)}M^I_{12})/Z$	${}^{(4)}M^I_{12}/Z$	M^I_{12}/Z
0	−5.0 i 0	18.2 − i 0	−6.1 −i 0	+14.1 −i 0
105	− 5.9 −i 1.7	21.1 − i 1.8	−5.9 −i.34	18.6 −i3.4
194	−3.8 −i 2.7	16.2 − i 1.0	− 5.1 −i 1.1	14.6 −i9.5

Now

The results for M are given in Table III.

TABLE III
Values of the matrix element M

q_L (MeV/c)	M
0	$t_1^0 + t_2^0 - Z\,(13.3 - i\,0)$
105	$t_1^0 + t_2^0 - Z\,(2.2 - i16.3)$
194	$t_1^0 + t_2^0 - Z\,(4.4 - i23.3)$

<u>Charge Exchange</u>. In the plane wave approximation, the total contribution due to charge exchange scattering is given in Table IV.

TABLE IV

Values of the matrix element M_{12}^{ex}.

q_L (Mev/c)	M_{12}^{ex}/z'
0	$-(5.0 + i\ 0)$
105	$-(5.9 + i\ 1.7)$
19[illegible]	$-(3.8 + i\ 2.7)$

where $z' = \frac{\mu_d}{\pi^2} \cdot N^2 t_1^{o\,ex} \cdot t_2^{o\,ex}$

Now total matrix element is

$$M^T = M + M_{12}^{ex}$$

which is given in Table V obtained from Tables III and IV.

Table V

Values of the matrix element M^T.

q_L (Mev/c)	M^T
0	$t_1^o + t_2^o - z(13.3 - i\ 0) - z'(5.0 + i\ 0)$
105	$t_1^o + t_1^o + z(2.2 - i16.3) - z'(5.9 + i1.7)$
194	$t_1^o + t_1^o + z(4.4 - i23.3) - z'(3.8 + i2.7)$

CHAPTER V

BRUECKNER MODEL

We have shown in Chapter I, that the scattering amplitude for the K^--meson is given by

$$F_\rho(\underline{q}',\underline{q}) = \left(1-\eta_p\eta_n \frac{e^{2iq\rho}}{\rho^2}\right)^{-1} \left[\eta_p\, e^{i(\underline{q}-\underline{q}')\cdot\underline{R}_1} + \eta_n e^{i(\underline{q}-\underline{q}')\cdot\underline{R}_2} + \eta_p\eta_n \frac{e^{iq\rho}}{\rho}\left(e^{i(\underline{q}\cdot\underline{R}_1-\underline{q}'\cdot\underline{R}_2)} + e^{i(\underline{q}\cdot\underline{R}_2-\underline{q}'\cdot\underline{R}_1)}\right)\right] \tag{I.4}$$

where

$$\rho = |\underline{R}_1 - \underline{R}_2|$$

We can write

$$\underline{R}_1 = \rho/2 \quad \text{and} \quad \underline{R}_2 = -\rho/2\,.$$

We get

$$F_\rho(\underline{q}',\underline{q}) = \frac{1}{D}\left[\eta_p\, e^{i(\underline{q}-\underline{q}')\cdot\rho/2} + e^{-i(\underline{q}-\underline{q}')\cdot\rho/2} + \eta_p\eta_n \frac{e^{iq\rho}}{\rho}\left(e^{i(\underline{q}+\underline{q}')\cdot\rho/2} + e^{-i(q+q')\cdot\rho/2}\right)\right] \tag{5.1}$$

where[19]

$$\mathcal{D} = \left(1 - \eta_p \eta_n \, e^{2iq\rho}/\rho^2\right)^{-1} \tag{5.2}$$

Since the scattering of the K^--meson from the deuteron can take place whatever the separation ρ and the direction of ρ may be, we have to average over ρ and the direction of $\underline{\rho}$. Anticipating the spherically symmetrical wave function for the deuteron, we can average over the angles between the ρ and q, and ρ and $\underline{q}'$, we get

$$f(\theta,\rho) = N/\mathcal{D} \tag{5.3}$$

where

$$N = (\eta_p + \eta_n) \frac{\sin p\rho/2}{p\rho/2} + 2\eta_p \eta_n \frac{e^{iq\rho}}{\rho} \frac{\sin P\rho}{P\rho} \tag{5.4}$$

$$p = 2q \sin \theta/2 \quad , \quad P = q \cos \theta/2 \tag{5.5}$$

θ is the angle of scattering of the meson in the K^--d center of mass system.

Here, $\underline{q}$ is the final K^- momentum in the K^--d center of mass system. In the above equations, η_p and η_n are the K^--P and K^--n scattering amplitudes (in the deuteron) and are related to the free

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19 See ref. 11

K^--nucleon amplitudes (f_p and f_n) by

$$\eta_p = \mu_{Kd}/\mu_{Kp}\, f_p \qquad , \quad \eta_n = \frac{\mu_{Kd}}{\mu_{Kn}}\, f_n \tag{5.6}$$

Where μ's are appropriate reduced masses, f_p and f_n are given in terms of the scattering lengths A_0 and A_1 by

$$\left.\begin{aligned} f_p &= \frac{1}{2}\left(\frac{A_1}{1 - i q_0 A_1} + \frac{A_0}{1 - i q_0 A_0}\right) \\ f_n &= \frac{A_1}{1 - i q_0 A_1} \end{aligned}\right] \tag{5.7}$$

q_0 is the K^--momentum in the K^-- nucleon center of mass system.

$$q_0 = \frac{m_p}{m_k + m_p}\, q_L \tag{5.8}$$

where q_L is the K^--momentum in the laboratory system

The scattering amplitude from a deuteron is given by averaging over all ρ. Therefore, the scattering amplitude is

$$f(\theta) = \langle f(\theta,\rho)\rangle = \int |\Phi(\rho)|^2 f(\theta,\rho)\, d\rho \tag{5.9}$$

We consider the contributions up to double scattering only.

Double scattering approximation.

$$f(\theta) = f_{imp}(\theta) + 2\eta_p \eta_n \left\langle \frac{e^{iq\rho}}{\rho} \frac{\sin p\rho}{p\rho} \right\rangle$$

$$= f_{imp}(\theta) + 2\eta_p \eta_n \cdot g(\theta) \tag{5.10}$$

where

$$f_{imp}(\theta) = (\eta_p + \eta_n) \left\langle \frac{\sin(p\rho/2)}{p\rho/2} \right\rangle \tag{5.11}$$

$$g(\theta) = \mathrm{Re}\, g(\theta) + i\, \mathrm{Im}\, g(\theta)$$

$$\mathrm{Re}\, g(\theta) = \frac{2\alpha}{p(1-\alpha a)} \left[-\alpha \tan^{-1}\frac{p+q}{2\alpha} - \beta \tan^{-1}\frac{p+q}{2\beta} + (\alpha+\beta) \times \right.$$

$$\times \tan^{-1}\frac{p+q}{\alpha+\beta} + \frac{p+q}{4} \log_e \left| \frac{\left((\alpha+\beta)^2 + (p+q)^2\right)^2}{\left(4\alpha^2 + (p+q)^2\right)\left(4\beta^2 + (p+q)^2\right)} \right|$$

$$\left. + \text{ similar terms with } q \to -q \right] \tag{5.12}$$

$$\operatorname{Im} g(\theta) = \frac{2\alpha}{p(1-\alpha a)} \Bigg[-\frac{\alpha}{2} \log_e \left| \frac{4\alpha^2 + (p+q)^2}{4\alpha^2 + (p-q)^2} \right| + q\left(\frac{1}{2} \tan^{-1} \frac{4\alpha p}{4\alpha^2 - p^2 + q^2} + \right.$$

$$\left. \epsilon(2\alpha, p, q) \cdot \pi/2 \right) + p\left(\frac{1}{2} \tan^{-1} \frac{4\alpha q}{4\alpha^2 - q^2 + p^2} + \epsilon(2\alpha, q, p)\pi/2 \right)$$

$$+ \text{ similar terms with } \alpha \to \beta \;\; -2(\text{ similar terms with } \alpha \to \tfrac{\alpha+\beta}{2}) \Bigg] \tag{5.13}$$

In Eq. (5.13)

$$\left. \begin{aligned} \epsilon(a, b, c) &= 1 \qquad , \quad a^2 - b^2 + c^2 < 0 \\ &= 0 \qquad , \quad a^2 - b^2 + c^2 \geqslant 0 \end{aligned} \right] \tag{5.14}$$

The values of $g(0^\circ)$ are given in Table VI.

TABLE VI
Values of the double scattering contribution g(0)

q_L (Mev/c)	$g(0^\circ)$
0	.57 + i 0
105	.38 + i .24
194	.21 + i .27

CHAPTER VI

A. COMPARISON OF BRUECKNER MODEL WITH THE MODEL GIVEN IN CHAPTER III

To compare the two models we should express t_1° and t_2° in terms of η_p and η_n.

Consider the scattering of the K^--meson from proton. The scattering amplitude is given by

$$f_p = -\frac{\mu_{KP}}{2\pi} \langle \underline{q}' | t_1 | \underline{q} \rangle$$

where $\underline{q}$ and $\underline{q}'$ are the momenta of the K^--meson in the center of mass system of K^--P system.

$$t_1 = t_1^\circ \delta(\underline{r} - \underline{\rho}/2) = t_1^\circ \delta(\underline{r}_0 - \underline{r}_1) \qquad \text{(Using Eq.(2.3))}$$

Therefore,

$$f_p = -\frac{\mu_{KP}}{2\pi} t_1^\circ \langle q' | \delta(\underline{r}_0 - \underline{r}_1) | q \rangle$$

$$= -\mu_{KP}/2\pi \; t_1^\circ \qquad (6.1)$$

If the proton is bound in the deuteron, then the effective amplitude of scattering is

$$\eta_p = \mu_{KD}/\mu_{KP} \cdot f_p \qquad (5.6)$$

since the amplitude f_p is proportioned to the reduced mass of the scattering particles, we get

$$\eta_p = - \frac{\mu_{Kp}}{2\pi} t_1^0$$

or
$$t_1^0 = - \frac{2\pi}{\mu_{Kp}} \eta_p \qquad (6.2)$$

Similarly

$$t_2^0 = - \frac{2\pi}{\mu_{Kn}} \cdot \eta_n \qquad (6.3)$$

Substituting Eqs. (6.2) and (6.3) in Eq. (3.105), we get

$$Z = \mu_{d}/\pi^2 N^2 t_1^0 t_2^0$$

$$= \frac{2 \mu_d N^2}{\mu_{Kp} \mu_{Kn}} \cdot (2 \eta_p \eta_n) \qquad (6.4)$$

The scattering amplitude of the K^- d scattering is defined by

$$f_{Kd} = - \frac{\mu_{Kd}}{2\pi} M \qquad (6.5)$$

Considering only the double scattering without charge exchange in the intermediate state, the scattering amplitude is given by

$$f_{Kd}^d = - \frac{\mu_{Kd}}{2\pi} \left(M_{12}^C + M_{12}^I \right) \qquad (6.6)$$

Making use of the results for M_{12}^{C} and M_{12}^{I} given in Tables I and II in Eqs. (6.4) and (6.6), we can find the values of $f_{Kd}^{d}/2\eta_p\eta_n$ at various energies. These values are given in Table VII along with the values of $\mathfrak{g}(0)$ (from Table VI) which represents the corresponding expression in the Brueckner model. Neglecting the continuum state contribution, we calculate the scattering amplitude $f_{Kd}^{\prime d}/2\eta_p\eta_n$ for double scattering due to the bound state of the deuteron. These values are also given in Table VII.

TABLE VII.

Comparison of the double scattering contribution given by Brueckner model and the model given in Chapter III.

q_L $\left(\frac{\mathrm{Mev}}{c}\right)$	$f_{Kd}^{\prime d}/2\eta_p\eta_n$	$f_{Kd}^{d}/2\eta_p\eta_n$	$\mathfrak{g}(0^\circ)$
0	.93 + i 0	.45 + i 0	.57 + i 0
105	.56 + i.44	-.07 + i.55	.38 + i.24
194	.36 + i.47	-.15 + i.79	.21 + i.27

We conclude, on comparing the 3rd and 4th columns, that the Brueckner model is quite unreliable for $q \gtrsim 100\ \mathrm{Mev}/c$, especially for what would be the real part of f_{Kd}^{d} if η_p and η_n were real. Comparison with the 2nd column shows that in some sense the Brueckner model approximates the effect of the continuum states very badly, agreeing much better with just the bound state contribution. This result seems reasonable from a physical point of view

B. CORRESPONDENCE BETWEEN THE BRUECKNER MODEL AND THE MODEL GIVEN IN CHAPTER II.

We can deduce the Brueckner model from the model given in Chapter II. The K^--d scattering amplitude is given by

$$f_{Kd} = -\frac{\mu_{Kd}}{2\pi} \langle \underline{q}', \varphi | t_1 + t_2 + t_1 g t_2 + t_2 g t_1 + \cdots | \varphi, \underline{q} \rangle \tag{6.7}$$

Using Eq. (3.1), we get

$$\langle \underline{q}', \varphi | t_1 | \varphi, \underline{q} \rangle = \int e^{i(\underline{q}-\underline{q}')\cdot \underline{r}} |\varphi(\rho)|^2 t_1^0 \, \delta(\underline{r} - \rho/2) \, d\underline{r} \, d\rho$$

$$= \langle \varphi | t_1^0 e^{i\underline{Q}\cdot\rho} | \varphi \rangle \tag{6.8}$$

where is defined in Eq. (3.34).

Similarly

$$\langle \underline{q}', \varphi | t_2 | \varphi, \underline{q} \rangle = \langle \varphi | t_2^0 e^{-i\underline{Q}\cdot\rho} | \varphi \rangle \tag{6.9}$$

Now[20]

$$\langle \underline{q}', \varphi(\rho) | t_1 g t_2 | \varphi(\rho), \underline{q} \rangle = \iint \frac{d\underline{l}\, d\underline{t}}{(2\pi)^6} \langle \underline{q}', \varphi(\rho) | t_1 | \underline{l}, \underline{t} \rangle \times$$

$$\left(q^2/2\mu_{Kd} + \epsilon_B - E_l - E_t + i\epsilon \right)^{-1} \langle \underline{t}, \underline{l} | t_2 | \varphi, \underline{q} \rangle \tag{6.10}$$

where E_l and E_t are defined by Eqs. (3.49 and (3.50).

Let us suppose that E_t is replaced by some average value $\bar{E}_t$ and

$$q^2/2\mu_{Kd} + \epsilon_B - \bar{E}_t = \bar{q}^2/2\mu_{Kd}. \tag{6.11}$$

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20 Taking U = 0

Using Eq. (6.11) in Eq. (6.10), we get

$$\langle \underline{q}', \varphi | t_1 \mathcal{G} t_2 | \varphi, \underline{q} \rangle = \iint \frac{d\underline{\ell}\, d\underline{t}}{(2\pi)^6} \langle \underline{q}', \varphi | t_1 | \underline{\ell}, \underline{t} \rangle \left(\bar{q}^2/2\mu_{\kappa d} - E_\ell + i\epsilon \right)^{-1} \times$$

$$\langle \underline{t}, \underline{\ell} | t_2 | \varphi, q \rangle .$$

$$= \int \frac{d\underline{\ell}}{(2\pi)^3} \langle \underline{q}', \varphi | t_1 | \underline{\ell} \rangle \left(\bar{q}^2/2\mu_{\kappa d} - E_\ell + i\epsilon \right)^{-1} \times$$

$$\delta(\rho - \rho') , \langle \underline{\ell} | t_2 | \varphi, \underline{q} \rangle$$

Using

$$\int \frac{d\underline{\ell}}{(2\pi)^3} \frac{|\underline{\ell}\rangle\langle \underline{\ell}|}{(\bar{q}^2/2\mu_{\kappa d} - E_\ell + i\epsilon)} = -\frac{\mu_{\kappa d}}{2\pi} \frac{e^{i\bar{q}|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|} ,$$

we get

$$\langle \underline{q}', \varphi | t_1 \mathcal{G} t_2 | \varphi\, \underline{q} \rangle = \iiiint d\underline{r}\, d\rho\, d\underline{r}'\, d\rho' \Big\{ e^{-i\underline{q}' \cdot \underline{r}} \varphi(\rho)\, t_1^0 \delta(\underline{r} - \rho/2)$$

$$\left(-\frac{\mu_{\kappa d}}{2\pi}\right) \frac{e^{i\bar{q}(\underline{r} - \underline{r}')}}{|\underline{r} - \underline{r}'|} \delta(\rho - \rho') \times$$

$$t_2\, \delta(\underline{r} + \rho/2)\, \varphi(\rho')\, e^{i\underline{q} \cdot \underline{r}} \Big\}$$

$$= \int d\rho\; e^{i(\underline{q} + \underline{q}') \cdot \rho/2} \left(-\frac{\mu_{\kappa d}}{2\pi} t_1^0 t_2^0 \frac{e^{i\bar{q}\rho}}{\rho} \right) |\varphi(\rho)|^2 .$$

$$= \left\langle \varphi(\rho) \right| e^{i(\underline{q} + \underline{q}') \cdot \rho/2} \left(-\frac{\mu_{\kappa d}}{2\pi} t_1^0 t_2^0 \frac{e^{i\bar{q}\rho}}{\rho} \right) \left| \varphi(\rho) \right\rangle \qquad (6.12)$$

Similarly

$$\langle \underline{q}', \varphi | t_2 \mathcal{G} t_1 | \varphi, \underline{q} \rangle = \langle \varphi | e^{-i(\underline{q}+\underline{q}')\cdot \rho/2} \left(-\frac{\mu_{Kd}}{2\pi} t_1^\circ t_2^\circ \frac{e^{i\bar{q}\rho}}{\rho} \right) | \varphi \rangle \tag{6.13}$$

Substituting Eqs. (6.8), (6.9), (6.12) and (6.13) in Eq. (6.7), we get

$$f_{Kd} = -\frac{\mu_{Kd}}{2\pi} \langle \varphi | t_1^\circ e^{i\underline{Q}\cdot\rho} + t_2^\circ e^{-i\underline{Q}\cdot\rho} - \frac{\mu_{Kd}}{2\pi} t_1^\circ t_2^\circ \frac{e^{i\bar{q}\rho}}{\rho} \times \left(e^{i(\underline{q}+\underline{q}')\cdot\rho/2} + e^{-i(\underline{q}+\underline{q}')\cdot\rho/2} \right) | \varphi \rangle \tag{6.14}$$

Using Eqs. (6.1) and (5.6), we get

$$-\frac{\mu_{Kd}}{2\pi} t_1^\circ = \eta_p \tag{6.15}$$

and

$$-\frac{\mu_{Kd}}{2\pi} t_2^\circ = \eta_n \tag{6.16}$$

Substituting Eqs. (6.15) and (6.16) in Eq. (6.14), we get

$$f_{Kd} = \langle \varphi(\rho) | \eta_p e^{i\underline{Q}\cdot\rho} + \eta_n e^{-i\underline{Q}\cdot\rho} + \eta_p \eta_n \frac{e^{i\bar{q}\rho}}{\rho} \times \left(e^{i(\underline{q}+\underline{q}')\cdot\rho/2} + e^{-i(\underline{q}+\underline{q}')\cdot\rho/2} \right) | \varphi(\rho) \rangle \tag{6.17}$$

It is obvious that Eq. (6.17) is identical to the scattering amplitude obtained by using the Brueckner model except q is replaced by $\bar{q}$. If $\bar{E}_t = \epsilon_B$, we get $\bar{q} = q$ and two models give the same scattering amplitudes. It seems likely that some choice of $\bar{q}$, such that $\bar{q}/q \neq 1$, might give better agreement with the results of our model calculation (See Table VII, Columns 3rd and 4th). This would be interesting to pursue a further investigation.

CHAPTER VII

COMPARISON OF THE MODEL GIVEN IN CHAPTER III WITH THE EXPERIMENTAL DATA.

A. Computation of Cross Sections. In this chapter, we shall try to compute the differential, elastic and total cross sections. As mentioned before in Chapter III, the matrix elements for any angle between q and q' reduce to the triple integrals which are hard to evaluate, the differential cross section is available only for the forward scattering. Consequently a rough estimation of the elastic cross section is obtained by using the relation

$$\sigma_{el} \sim \frac{(d\sigma/d\Omega)_{\theta=0^\circ} \int (d\sigma_{imp}/d\Omega)\, d\Omega}{(d\sigma_{imp}/d\Omega)_{\theta=0^\circ}} \tag{7.1}$$

where $\frac{d\sigma_{imp}}{d\Omega}$ is the differential cross section in the impulse approximation.

The total cross section is calculated by using the optical theorem

$$\sigma_T = \frac{4\pi}{q} \cdot \mathrm{Im}\, f(0^\circ) \tag{7.2}$$

The scattering lengths f_p and f_n defined by Eq. (5.7) are calculated by using the Dalitz Solutions[21] for the scattering lengths in the zero range approximation. The scattering lengths are:

Sol. I $\quad A_1 = .02 + i\ .38 \quad f$

$\qquad\quad A_0 = -.22 + i\ .?.74 \quad f$

Sol. II $\quad A_1 = 1.20 + i\ .56 \quad f$

$\qquad\quad A_0 = -.59 + i\ .96 \quad f$

The values of η_p and η_n defined by Eq. (5.6) are given in Table VIII.

TABLE VIII.

Values of η_p and η_n.

q_L ($\frac{Mev}{c}$)	Sol.	$\eta_p (f)$	$\eta_n (f)$
105	I	$-.03 + i\ 1.05$	$.02 + i\ .41$
	II	$.26 + i\ .91$	$.90 + i\ .88$
194	I	$-.01 + i\ .78$	$.02 + i\ .37$
	II	$.17 + i\ .81$	$.59 + i\ .83$

21 See refs. 14 and 15.

Using the definitions of t_1^0 and t_2^0 given in Eqs. (6.2) and (6.3) and the results given in Tables III and VIII, we can find the scattering amplitudes $f_{K\Lambda}$. The results are given in Table IX.

TABLE IX.

Numerical values of the scattering amplitudes.

q_L (Mev/c)	Sol.	$f_{K\Lambda}$ (f)
105	I	.05 + i 1.29
	II	.33 + i 1.38
194	I	.08 + i .93
	II	.12 + i .89

The differential, elastic and total cross sections are given in[22] Table X. The elastic cross section is calculated by using Eq. (7.1).

TABLE X.

Values of the elastic and total cross sections.

q_L (Mev/c)	Sol.	$\frac{d\sigma}{d\Omega}(0°)$ mb/St	σ_{el} mb	σ_T mb
105	I	16.7	110	383
	II	20.1	132	410
194	I	8.7	35	150
	II	8.0	31	142

22 As the coulomb interaction is significant at low energies, the cross sections are not calculated at 0 Mev/c .

B. <u>Contribution Due to Charge Exchange</u>. The scattering amplitude for the process

$$K^- + p \to \bar{K}^0 + n$$

is given by

$$f_p^{ex} = \tfrac{1}{2}\left[\frac{A_1}{1 - iq_0 A_1} - \frac{A_0}{1 - iq_0 A_0}\right]$$

It is obvious that for the inverse process

$$\bar{K}^0 + n \to K^- + p$$

we have

$$f_n^{ex} = f_p^{ex}$$

The corresponding expressions for η_p^{ex}, η_n^{ex} are given by Eq.(5.6) with f_p and f_n replaced by f_p^{ex} and f_n^{ex}.

The order of magnitude of the contribution due to charge exchange in the intermediate state can be estimated easily. We can write the elastic cross section as

$$\sigma_{el} \sim \left| M_{imp} + M_d^{ord} + M_d^{ex} \right|^2$$

$$= \left| M_{imp} + M_d^{ord} \right|^2 \left[1 + 2\,\mathrm{Re}\left(\frac{M_d^{ex}}{M_{imp} + M_d^{ord}} \right) + \cdots \right] \qquad (7.3)$$

where $$M_d^{ord} = M_{12}^{C} + M_{12}^{I}$$

and $$M_d^{ex} = M_{12}^{ex}$$

Using plane wave approximation in the evaluations of M_{12}^{ex} the values of 2 Re $\dfrac{M_d^{ex}}{M_{imp} + M_d^{ord}}$ at $\theta = 0^{o}$ are calculated by using the Tables III and IV. The results are given in Table XI.

TABLE XI.

Ratio of the charge exchange scattering to ordinary scattering.

q_L $\frac{Mev}{c}$	Sol.	2 Re $\frac{M_d^{ex}}{M_{imp} + M_d^{ord}}$
105	I	− .24
	II	.25
194	I	− .17
	II	.18

It is obvious from Table XI, that the contribution due to charge exchange in the intermediate state is small compared to the ordinary scattering contribution. However, if anything approaching say 20% accuracy in computing cross sections is desired, these effects would have to be included.

If we take the plane wave approximation in the charge exchange contribution, we can calculate the cross sections using the Table V. The results are given in Table XII.

TABLE XII.

The cross sections including virtual charge exchange.

q_L (Mev/c)	Sol.	$f_{Kd}(+)$	$\frac{d\sigma}{d\Omega}(0°)$	σ_{el}	σ_T
			mb/st	mb	mb
105	I	−.28 + i 1.15	14.0	92	342
	II	.82 + i 1.44	27.5	180	428
194	I	−.01 + i .86	7.4	29	138
	II	.21 + i .36	9.7	38	154

C. **Break up Cross Sections and Comparison with the Experimental Results.** A crude estimation of the absorption cross section of the K^--nucleon is made and the results are given in Tables XIII. We shall take

$$\sigma_d^{ab} \sim \sigma_p^{ab} + \sigma_n^{ab}$$

where

$$\sigma_p^{ab} = \sigma_p^{total} - \sigma_p^{el}$$

and

$$\sigma_n^{ab} = \sigma_n^{total} - \sigma_n^{el}$$

Using the Table XII, the results for $(\sigma_T - \sigma_d^{ab})$ are also given in Table XIII.

TABLE XIII.

Values of $(\sigma_T - \sigma_d^{ab})$.

$q_L\left(\frac{Mev}{c}\right)$	Sol.	σ_p^{ab} mb	σ_n^{ab} mb	σ_d^{ab} mb	$\sigma_T - \sigma_d^{ab}$ mb
105	I	181	106	287	55
	II	155	125	280	148
194	I	67	48	115	23
	II	66	44	110	44

We have

$$\sigma_T = \sigma_{el}(K^- + d \to K^- + d) + \sigma(K^- + d \to K^- + n + p) + \sigma(K^- + d \to \bar{K}^0 + n + n)$$

$$+ \sigma(\text{Hyperon production})$$

$$= \sigma_{el}(K^- + d \to K^- + d) + \sigma(K^- + d \to K^- + n + p) + \sigma_d^{ab}$$

Therefore

$$\sigma_T - \sigma_d^{ab} = \sigma_{el}(K^- + d \to K^- + d) + \sigma(K^- + d \to K^- + n + p) \qquad (7.4)$$

The experimental data[23] for the total cross sections for the reactions $K^- + d \to K^- + d$

and $K^- + d \to K^- + n + p$

is given in Table XIV.

TABLE XIV.

The experimental data and the theoretical values for the total cross sections for the reactions $K^- + d \to K^- + d$ and $K^- + d \to K^- + n + p$

q_L (Mev/c)	$\sigma_{exp.}$ (mb)	$\sigma_T - \sigma_d^{el}$ Sol. I (mb)	Sol. II (mb)
105		55	148
125	145 ± 35		
175	55 ± 15		
194		23	44
210	95 ± 25		

23 L. Alvarez, Proceedings of the Ninth International Annual Conference on High Energy Physics (Academy of Sciences, U.S.S.R., 1960) p. 471

D. Conclusions. A study of the Table XIV shows that the Dalitz Solution II is more favorable compared to the Solution I.

It should be emphasized that some of the estimates made to get the comparison with the data have been extremely crude. They have been made only for illustration purposes. A more serious test and application of the model in question would require the computations for other angles than 0° of the elastic amplitude including a more careful estimate of the virtual charge exchange - and this could certainly be done by machine calculation - as well as a direct computation of the break-up cross section $\sigma(K^- + d \rightarrow K^- + n + p)$ within the framework of this model. Again, this seems feasible and the results obtained in this thesis would seem to justify further work along this line Our conclusion that Solution II is favored by the deuterium data is the same as that reached recently by Chand[24] and Dalitz who have studied the same problem by an entirely different method which is, however, closely related to the Brueckner model, binding and recoil corrections being neglected. Our results (Table VII) for the double scattering part of the amplitude in a model in which such corrections have been included do not support the conjecture of Chand[25] that corrections of this kind are unlikely to be important.

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24 Private communication.

25 R. Chand: Thesis on K^--Deuterium Interactions - University of Chicago.

APPENDIX

Evaluation of the integral (3.89)

We shall evaluate the integral

$$I = \int_0^{\infty} \frac{dy}{(y+\delta-i)^2 + \ell^2/4}$$

Since

$$\frac{1}{(y+\delta-i)^2+\ell^2/4} = \frac{1}{(y+\delta-i)^2-(i\ell/2)^2}$$

$$= \frac{1}{i\ell}\left[\frac{1}{y+\delta-i(1+\ell/2)} - \frac{1}{y+\delta-i(1-\ell/2)}\right]$$

Therefore, we get

$$\int_0^{\infty} \frac{dy}{y+\delta-i(1+\ell/2)} = \int_0^{\infty} \frac{y+\delta+i(1+\ell/2)}{(y+\delta)^2+(1+\ell/2)^2}$$

$$= \left[\tfrac{1}{2}\log\left|(y+\delta)^2+(1+\ell/2)^2\right| + i\,\frac{(1+\ell/2)}{(1+\ell/2)}\tan^{-1}\left|\frac{y+\delta}{1+\ell/2}\right|\right]_0^{\infty}$$

$$= \left[\tfrac{1}{2}\log\left|(y+\delta)^2+(1+\ell/2)^2\right|\right]_0^{\infty} + i\left(\pi/2 - \tan^{-1}\left|\delta/1+\ell/2\right|\right) . \qquad \textbf{(A.1)}$$

For $\ell < 2$, we get

$$\int_0^\infty \frac{dy}{(y+\delta) - i(1-\ell/2)} = \int_0^\infty \frac{(y+\delta) + i(1-\ell/2)}{(y+\delta)^2 + (1-\ell/2)^2}\, dy$$

$$= \left[\tfrac{1}{2} \log_e \left|(y+\delta)^2 + (1-\ell/2)^2\right| + \frac{i(1-\ell/2)}{(1-\ell/2)} \times \tan^{-1}\left|\frac{y+\delta}{1-\ell/2}\right|\right]_0^\infty$$

$$= \left[\tfrac{1}{2}\log_e\left|(y+\delta)^2 + (1-\ell/2)^2\right|\right]_0^\infty + i\left(\pi/2 - \tan^{-1}\left|\delta/1-\ell/2\right|\right). \qquad \textbf{(A.2)}$$

From Eqs. (A.1) and (A.2), we get for $\ell < 2$

$$\int_0^\infty \frac{dy}{(y+\delta)^2 + \ell^2/4} = \frac{1}{i\ell}\left[-\tfrac{1}{2}\log_e\left|\frac{(1+\ell/2)^2+\delta^2}{(1-\ell/2)^2+\delta^2}\right| + i\left(\tan^{-1}\left|\frac{\delta}{1-\ell/2}\right| - \tan^{-1}\left|\frac{\delta}{1+\ell/2}\right|\right)\right.$$

$$= \frac{i}{2\ell}\log_e\left|\frac{(1+\ell/2)^2+\delta^2}{(1-\ell/2)^2+\delta^2}\right| + \frac{T(\delta,\ell)}{\ell} \qquad (3.89)$$

For $\ell > 2$, we get

$$\int_0^\infty \frac{dy}{(y+\delta) - i(1-\ell/2)} = \int_0^\infty \frac{(y+\delta) - i(\ell/2-1)}{(y+\delta)^2 + (\ell/2-1)^2}$$

$$= \left[\tfrac{1}{2}\log_e\left|(y+\delta)^2 + (\ell/2-1)^2\right| + i\,\frac{(\ell/2-1)}{(\ell/2+1)}\,\tan^{-1}\left|\frac{y+\delta}{\ell/2-1}\right|\right]_0^\infty$$

$$= \left[\tfrac{1}{2}\log_e\left|(y+\delta)^2 + (\ell/2-1)^2\right|\right]_0^\infty - i\left(\pi/2 - \tan^{-1}\left|\delta/\ell/2-1\right|\right). \qquad \textbf{(A.3)}$$

From Eqs. (A.1) and (A.3), we get for $\ell > 2$

$$\int_0^\infty \frac{dy}{(y+\delta-i)^2 + \ell^2/4} = \frac{1}{i\ell}\left[-\frac{1}{2}\log\left|\frac{(1+\ell/2)^2+\delta^2}{(1-\ell/2)^2+\delta^2}\right| + \frac{1}{\ell}\left(\pi - \tan^{-1}\left|\frac{\delta}{1+\ell/2}\right| - \tan^{-1}\left|\frac{\delta}{1-\ell/2}\right|\right)\right]$$

$$= \frac{i}{2\ell}\log_e\left|\frac{(1+\ell/2)^2+\delta^2}{(1-\ell/2)^2+\delta^2}\right| + \frac{T(\delta,\ell)}{\ell} \qquad (3.89)$$

where T (δ, ℓ) is defined in Eqs. (3.90a) and (3.90b)

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